Math 227 Problem Set IX
Due Friday, April 1

1. Evaluate, both by direct integration and by Stokes' Theorem, $\oint_C (z\,dx + x\,dy + y\,dz)$ where C is the circle $x + y + z = 0$, $x^2 + y^2 + z^2 = 1$. Orient C so that its projection on the xy–plane is counterclockwise.

2. Let C be the intersection of $x + 2y - z = 7$ and $x^2 - 2x + 4y^2 = 15$. The curve C is oriented counterclockwise when viewed from high on the z–axis. Let

$$\mathbf{F} = \big(e^{x^2} + yz\big)\hat{\boldsymbol{\imath}} + \big(\cos(y^2) - x^2\big)\hat{\boldsymbol{\jmath}} + \big(\sin(z^2) + xy\big)\hat{\mathbf{k}}$$

   Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$.

3. Consider $\iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS$ where S is the portion of the sphere $x^2 + y^2 + z^2 = 1$ that obeys $x+y+z \geq 1$, $\hat{\mathbf{n}}$ is the upward pointing normal to the sphere and $\mathbf{F} = (y-z)\hat{\boldsymbol{\imath}} + (z-x)\hat{\boldsymbol{\jmath}} + (x-y)\hat{\mathbf{k}}$. Find another surface S' with the property that $\iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS = \iint_{S'} (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS$ and evaluate $\iint_{S'} (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS$.

4. Verify the identity $\oint_C \phi \boldsymbol{\nabla}\psi \cdot d\mathbf{r} = -\oint_C \psi \boldsymbol{\nabla}\phi \cdot d\mathbf{r}$ for any continuously differentiable scalar fields ϕ and ψ and curve C that is the boundary of a piecewise smooth surface.

5. Use Green's Theorem to show that

$$\iint_R \boldsymbol{\nabla} \cdot \mathbf{F}\; dA = \oint_{\partial R} \mathbf{F} \cdot \hat{\mathbf{n}}\, ds$$

   where ∂R is the boundary of the plane domain R, with the usual orientation, $\mathbf{F} = F_1\hat{\boldsymbol{\imath}} + F_2\hat{\boldsymbol{\jmath}}$, $\hat{\mathbf{n}}$ is the outward normal to ∂R and s is the arclength along ∂R.

6. Integrate $\frac{1}{2\pi} \oint_C \frac{x\,dy - y\,dx}{x^2+y^2}$ counterclockwise around
   (a) the circle $x^2 + y^2 = a^2$
   (b) the boundary of the square with vertices $(-1,-1)$, $(-1,1)$, $(1,1)$ and $(1,-1)$
   (c) the boundary of the region $1 \leq x^2 + y^2 \leq 2,\ y \geq 0$

7. Show that

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)$$

   for all $(x, y) \neq (0, 0)$. Discuss the connection between this result and the results of the last question.

8. Find a continuously differentiable simple, closed, counterclockwise oriented curve, C, in the xy–plane for the which the value of the line integral $\oint_C (y^3 - y)\,dx - 2x^3\,dy$ is a maximum among all C^1 simple, closed, counterclockwise oriented curves. "Simple" means that the curve does not intersect itself.

Reminder: The final exam is on Tuesday, April 19 at 12:00pm in room BUCH B315.