Math 227 Problem Set V

Due Wednesday, February 24

If $\mathcal{C}$ is a curve with parametrization $\mathbf{r}(t)$, $a \le t \le b$, then we define the line integral

$$\int_{\mathcal{C}} f\, ds = \int_a^b f\big(\mathbf{r}(t)\big)\, \left|\tfrac{d\mathbf{r}}{dt}(t)\right| dt$$

1. Evaluate the line integral $\int_{\mathcal{C}} f(x,y,z)\, ds$ for
   (a) $f(x,y,z) = x\cos z$, $\mathcal{C}$ the curve with parametrization $\mathbf{r}(t) = t\hat{\boldsymbol{\imath}} + t^2\hat{\boldsymbol{\jmath}}$, $0 \le t \le 1$.
   (b) $f(x,y,z) = \frac{x+y}{y+z}$, $\mathcal{C}$ the curve with parametrization $\mathbf{r}(t) = \big(t, \frac{2}{3}t^{3/2}, t\big)$, $1 \le t \le 2$.

2. (a) Show that the integral $\int_{\mathcal{C}} f(x,y)\, ds$ along the curve $\mathcal{C}$ given in polar coordinates by $r = r(\theta)$, $\theta_1 \le \theta \le \theta_2$, is

$$\int_{\theta_1}^{\theta_2} f\big(r(\theta)\cos\theta, r(\theta)\sin\theta\big)\sqrt{r(\theta)^2 + \big(\tfrac{dr}{d\theta}(\theta)\big)^2}\, d\theta$$

   (b) Compute the arc length of $r = 1 + \cos\theta,\ 0 \le \theta \le 2\pi$.

3. Find the mass of a wire formed by the intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the plane $x + y + z = 0$ if the density at (x,y,z) is given by $\rho(x,y,z) = x^2$ grams per unit length of wire.

4. Evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ for
   (a) $\mathbf{F}(x,y) = xy\hat{\boldsymbol{\imath}} - x^2\hat{\boldsymbol{\jmath}}$ along $y = x^2$ from $(0,0)$ to $(1,1)$.
   (b) $\mathbf{F}(x,y,z) = (x-z)\hat{\boldsymbol{\imath}} + (y-z)\hat{\boldsymbol{\jmath}} - (x+y)\hat{\mathbf{k}}$ along the polygonal curve from $(0,0,0)$ to $(1,0,0)$ to $(1,1,0)$ to $(1,1,1)$.

5. Evaluate $\oint_{\mathcal{C}} x^2y^2\, dx + x^3y\, dy$ counterclockwise around the square with vertices $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$.

6. Let $a < b$ and $c < d$. Let $\mathbf{r} : [a,b] \to \mathbb{R}^n$ be a parametrization for some curve $\mathcal{C}$. Let $u : [c,d] \to [a,b]$ be increasing, differentiable and obey $u(c) = a$ and $u(d) = b$. Set $\mathbf{R}(t) = \mathbf{r}(u(t))$. It is another parametrization of $\mathcal{C}$.
   (a) Prove that

$$\int_a^b \mathbf{F}\big(\mathbf{r}(t)\big) \cdot \mathbf{r}\,'(t)\, dt = \int_c^d \mathbf{F}\big(\mathbf{R}(t)\big) \cdot \mathbf{R}'(t)\, dt$$

   (b) Set $c = 0$ and $d = 1$. Find a function $u : [c,d] \to [a,b]$ which is increasing, infinitely differentiable and obeys $u(c) = a$ and $u(d) = b$. So, it is always possible to parametrize a curve in such a way that the parameter runs from 0 to 1.

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(c) Set $c = 0$ and $d = 1$. Find a function $u : [c, d] \to [a, b]$ which is strictly increasing, infinitely differentiable and obeys $u(c) = a$, $u(d) = b$, $u'(c) = 0$ and $u'(d) = 0$. So, it is always possible to parametrize a curve in such a way that the parameter runs from 0 to 1 and the velocity is zero at both end points.

(d) When we assign a parametrization $\mathbf{r} : [a, b] \to \mathbb{R}^n$ to a curve, we are implicitly designating one of the two end points of the curve (namely $\mathbf{r}(a)$) as the beginning of the curve and the other end point (namely $\mathbf{r}(b)$) as the end of the curve. This is called assigning an orientation to the curve. Given a curve $\mathcal{C}$ with orientation determined by a parametrization $\mathbf{r} : [0, 1] \to \mathbb{R}^n$, define $-\mathcal{C}$ to be the curve with orientation determined by the parametrization $\mathbf{w}(t) = \mathbf{r}(1 - t)$. The beginning of $\mathcal{C}$ is the end of $-\mathcal{C}$ and vice versa. This called reversing the orientation. Show that

$$\int_{-\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = -\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

Notation: "$u : [c, d] \to [a, b]$" means that u is a function that is defined on $c \le t \le d$ and takes values between a and b.

7. Find the work $\int \mathbf{F} \cdot d\mathbf{r}$ done by the force field $\mathbf{F} = (x + y)\hat{\boldsymbol{\imath}} + (x - z)\hat{\boldsymbol{\jmath}} + (z - y)\hat{\mathbf{k}}$ in moving an object from $(1, 0, -1)$ to $(0, -2, 3)$ along any smooth curve.

8. Evaluate $\int_{\mathcal{C}} ye^{xy} \sin(y + z)\, dx + e^{xy}\big(x \sin(y + z) + \cos(y + z)\big)\, dy + e^{xy} \cos(y + z)\, dz$ along the straight line segment from $(0, 0, 0)$ to $\big(1, \frac{\pi}{4}, \frac{\pi}{4}\big)$.

9. Let $\mathbf{F}$ be a continous vector field on $\mathbb{R}^d$. Prove that the following two statements are equivalent.
   (i) If $\mathcal{C}_1$ and $\mathcal{C}_2$ are any two curves in $\mathbb{R}^d$ with the same initial and final points then $\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$.
   (ii) If $\mathcal{C}$ is any closed curve in $\mathbb{R}^d$ (i.e. its initial and final points are the same) then $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0$.

10. Let $\mathbf{F}$ be a continous vector field on $\mathbb{R}^d$. Fix any two points Q_1 and Q_2 in $\mathbb{R}^d$ and assume that $\int_{\mathcal{D}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{D}'} \mathbf{F} \cdot d\mathbf{r}$ for all curves $\mathcal{D}$ and $\mathcal{D}'$ that start at Q_1 and end at Q_2. Let P_1 and P_2 be any two points in $\mathbb{R}^d$. Prove that $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}'} \mathbf{F} \cdot d\mathbf{r}$ for all curves $\mathcal{C}$ and $\mathcal{C}'$ that start at P_1 and end at P_2.

11. Is $\mathbf{F} = y\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}} + y\hat{\mathbf{k}}$ a conservative field? Find infinitely many closed curves $\mathcal{C}$ for which $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0$.

Reminder: Midterm I is on Wednesday, February 10.

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