Math 227 Problem Set V
Due Wednesday, February 24

If $C$ is a curve with parametrization $r(t), a \leq t \leq b$, then we define the line integral

$$\int_C f \, ds = \int_a^b f(r(t)) \left| \frac{dr}{dt}(t) \right| \, dt$$

1. Evaluate the line integral $\int_C f(x, y, z) \, ds$ for
   (a) $f(x, y, z) = x \cos z$, $C$ the curve with parametrization $r(t) = t \hat{i} + t^2 \hat{j}$, $0 \leq t \leq 1$.
   (b) $f(x, y, z) = \frac{x+y}{y+z}$, $C$ the curve with parametrization $r(t) = (t, \frac{2}{3}t^{3/2}, t), 1 \leq t \leq 2$.

2. (a) Show that the integral $\int_C f(x, y) \, ds$ along the curve $C$ given in polar coordinates by $r = r(\theta), \theta_1 \leq \theta \leq \theta_2$, is

$$\int_{\theta_1}^{\theta_2} f(r(\theta) \cos \theta, r(\theta) \sin \theta) \sqrt{r(\theta)^2 + \left(\frac{dr}{d\theta}(\theta)\right)^2} \, d\theta$$

   (b) Compute the arc length of $r = 1 + \cos \theta$, $0 \leq \theta \leq 2\pi$.

3. Find the mass of a wire formed by the intersection of the sphere $x^2 + y^2 + z^2 = 1$ and the plane $x + y + z = 0$ if the density at $(x, y, z)$ is given by $\rho(x, y, z) = x^2$ grams per unit length of wire.

4. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for
   (a) $\mathbf{F}(x, y) = xy\hat{i} - x^2\hat{j}$ along $y = x^2$ from $(0, 0)$ to $(1, 1)$.
   (b) $\mathbf{F}(x, y, z) = (x - z)\hat{i} + (y - z)\hat{j} - (x + y)\hat{k}$ along the polygonal curve from $(0, 0, 0)$ to $(1, 0, 0)$ to $(1, 1, 0)$ to $(1, 1, 1)$.

5. Evaluate $\int_C x^2y^2 \, dx + x^3y \, dy$ counterclockwise around the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$.

6. Let $a < b$ and $c < d$. Let $r : [a, b] \to \mathbb{R}^n$ be a parametrization for some curve $C$. Let $u : [c, d] \to [a, b]$ be increasing, differentiable and obey $u(c) = a$ and $u(d) = b$. Set $\mathbf{R}(t) = r(u(t))$. It is another parametrization of $C$.
   (a) Prove that
   $$\int_a^b \mathbf{F}(r(t)) \cdot \mathbf{r}'(t) \, dt = \int_c^d \mathbf{F}(\mathbf{R}(t)) \cdot \mathbf{R}'(t) \, dt$$
   (b) Set $c = 0$ and $d = 1$. Find a function $u : [c, d] \to [a, b]$ which is increasing, infinitely differentiable and obeys $u(c) = a$ and $u(d) = b$. So, it is always possible to parametrize a curve in such a way that the parameter runs from 0 to 1.

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(c) Set \( c = 0 \) and \( d = 1 \). Find a function \( u : [c, d] \to [a, b] \) which is strictly increasing, infinitely differentiable and obeys \( u(c) = a, u(d) = b, u'(c) = 0 \) and \( u'(d) = 0 \). So, it is always possible to parametrize a curve in such a way that the parameter runs from 0 to 1 and the velocity is zero at both end points.

(d) When we assign a parametrization \( r : [a, b] \to \mathbb{R}^n \) to a curve, we are implicitly designating one of the two end points of the curve (namely \( r(a) \)) as the beginning of the curve and the other end point (namely \( r(b) \)) as the end of the curve. This is called assigning an orientation to the curve. Given a curve \( C \) with orientation determined by a parametrization \( r : [0, 1] \to \mathbb{R}^n \), define \(-C\) to be the curve with orientation determined by the parametrization \( w(t) = r(1 - t) \). The beginning of \( C \) is the end of \(-C\) and vice versa. This called reversing the orientation. Show that

\[ \int_{-C} F \cdot dr = -\int_C F \cdot dr \]

Notation: “\( u : [c, d] \to [a, b] \)” means that \( u \) is a function that is defined on \( c \leq t \leq d \) and takes values between \( a \) and \( b \).

7. Find the work \( \int F \cdot dr \) done by the force field \( F = (x + y)\mathbf{i} + (x - z)\mathbf{j} + (z - y)\mathbf{k} \) in moving an object from \((1, 0, -1)\) to \((0, -2, 3)\) along any smooth curve.

8. Evaluate \( \int_C ye^{xy} \sin(y + z) \, dx + e^{xy}(x \sin(y + z) + \cos(y + z)) \, dy + e^{xy} \cos(y + z) \, dz \) along the straight line segment from \((0, 0, 0)\) to \((1, \frac{\pi}{4}, \frac{\pi}{4})\).

9. Let \( F \) be a continuous vector field on \( \mathbb{R}^d \). Prove that the following two statements are equivalent.
   (i) If \( C_1 \) and \( C_2 \) are any two curves in \( \mathbb{R}^d \) with the same initial and final points then \( \int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr \).
   (ii) If \( C \) is any closed curve in \( \mathbb{R}^d \) (i.e. its initial and final points are the same) then \( \int_C F \cdot dr = 0 \).

10. Let \( F \) be a continuous vector field on \( \mathbb{R}^d \). Fix any two points \( Q_1 \) and \( Q_2 \) in \( \mathbb{R}^d \) and assume that \( \int_D F \cdot dr = \int_{D'} F \cdot dr \) for all curves \( D \) and \( D' \) that start at \( Q_1 \) and end at \( Q_2 \). Let \( P_1 \) and \( P_2 \) be any two points in \( \mathbb{R}^d \). Prove that \( \int_C F \cdot dr = \int_{C'} F \cdot dr \) for all curves \( C \) and \( C' \) that start at \( P_1 \) and end at \( P_2 \).

11. Is \( F = y\mathbf{i} + x\mathbf{j} + y\mathbf{k} \) a conservative field? Find infinitely many closed curves \( C \) for which \( \int_C F \cdot dr = 0 \).

Reminder: Midterm I is on Wednesday, February 10.