Math 227 Problem Set IV

Due Wednesday, February 3

1. For each of the following vector fields, sketch the vector field and determine its field lines.
   (a) $\mathbf{v}(x, y) = x\hat{\imath} + y\hat{\jmath}$.
   (b) $\mathbf{v}(x, y) = \nabla(x^2 - y)$.

2. Describe the stream lines of
   (a) $\mathbf{v}(x, y, z) = x\hat{\imath} + y\hat{\jmath} - x\hat{\mathbf{k}}$.
   (b) $\mathbf{v}(x, y, z) = \frac{x\hat{\imath}+y\hat{\jmath}}{(1+z^2)(x^2+y^2)}$.

3. Sketch the stream lines for the velocity field $\mathbf{v} = y\hat{\imath} - \sin x\hat{\jmath}$.

4. The vector fields $\hat{\mathbf{r}}(r, \theta) = \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}$ and $\hat{\boldsymbol{\theta}}(r, \theta) = -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}$ are often used as basis vectors when working with polar coordinates. (Since $\hat{\mathbf{r}}(r, \theta)$ and $\hat{\boldsymbol{\theta}}(r, \theta)$ happen to be independent of r, people usually just write $\hat{\mathbf{r}}(\theta)$, $\hat{\boldsymbol{\theta}}(\theta)$.) Show that the polar curve $r = e^\theta$ is a field line of the vector field $\mathbf{F}(r, \theta) = \hat{\mathbf{r}}(r, \theta) + \hat{\boldsymbol{\theta}}(r, \theta)$.

5. Let a particle of mass m move on a path $\mathbf{r}(t)$ according to Newton's law, $m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}$, in a force field $\mathbf{F} = -\nabla V$ on $\mathbb{R}^3$, where V is a given function (called the potential energy).
   (a) Prove that the energy $E(t) = \frac{1}{2}m|\mathbf{r}'(t)|^2 + V\big(\mathbf{r}(t)\big)$ is constant in time.
   (b) If the particle moves on an equipotential surface (i.e. a surface whose equation is $V(x, y, z) = V_0$, for some constant V_0), show that its speed is constant.

6. Let $f(\mathbf{x}, t)$ be a real–valued function of $\mathbf{x} \in \mathbb{R}^3$ and $t \in \mathbb{R}$. Define the **material derivative** of f relative to a vector field $\mathbf{F}(\mathbf{x})$ to be

$$\frac{Df}{Dt}(\mathbf{x}, t) = \frac{\partial f}{\partial t}(\mathbf{x}, t) + \nabla f(\mathbf{x}, t) \cdot \mathbf{F}(\mathbf{x})$$

   Here $\nabla f(\mathbf{x}, t)$ is the spatial gradient. That is,

$$\nabla f\big((x, y, z), t\big) = \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}$$

   Let $\mathbf{r}(t)$ be a flow line of $\mathbf{F}$. Show that

$$\frac{d}{dt}f\big(\mathbf{r}(t), t\big) = \frac{Df}{Dt}\big(\mathbf{r}(t), t\big)$$

   In words, $\frac{Df}{Dt}$ is the rate of change of f felt by a particle that is moving with the flow $\mathbf{F}$.

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7. Determine whether or not each of the following vector fields is conservative. If so, find a function ϕ so that $\mathbf{F} = \nabla\phi$.
   (a) $\mathbf{F}(x,y) = \frac{x\hat{\boldsymbol{\imath}} - y\hat{\boldsymbol{\jmath}}}{x^2+y^2}$
   (b) $\mathbf{F}(x,y) = \frac{y\hat{\boldsymbol{\imath}} - x\hat{\boldsymbol{\jmath}}}{x^2+y^2}$
   (c) $\mathbf{F}(x,y,z) = (2xy + z^2)\hat{\boldsymbol{\imath}} + (x^2 + 2yz)\hat{\boldsymbol{\jmath}} + (y^2 + 2xz)\hat{\mathbf{k}}$

8. Let $\mathbf{F}$ be a conservative force field with potential ϕ. Show that the lines of force are perpendicular to the equipotential surfaces (i.e. surfaces whose equations are of the form $\phi(x,y,z) = \phi_0$ with ϕ_0 constant).

9. Let $\psi(r,\theta)$ be a function in polar ccordinates. In cartesian coordinates, this function becomes $\phi(x,y) = \psi\big(r(x,y), \theta(x,y)\big)$ where $r(x,y) = \sqrt{x^2+y^2}$ and $\theta(x,y) = \tan^{-1}\frac{y}{x}$. By definition, the gradient $(\nabla\psi)(r,\theta)$ of $\psi(r,\theta)$ is defined by the requirement that

$$\nabla\phi(x,y) = (\nabla\psi)\big(r(x,y), \theta(x,y)\big)$$

   (a) Find $\nabla\psi(r,\theta)$, expressed in terms of the basis vectors

$$\hat{\mathbf{r}}(\theta) = \cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}} \qquad \hat{\boldsymbol{\theta}}(\theta) = -\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}$$

   (b) We have seen that $\frac{\partial F_1}{\partial y}(x,y) = \frac{\partial F_2}{\partial x}(x,y)$ is a necessary condition for the vector field $\mathbf{F}(x,y) = F_1(x,y)\,\hat{\boldsymbol{\imath}} + F_2(x,y)\,\hat{\boldsymbol{\jmath}}$ to be conservative. Find the analogous necessary condition for the vector field $\mathbf{G}(r,\theta) = G_1(r,\theta)\,\hat{\mathbf{r}}(\theta) + G_2(r,\theta)\,\hat{\boldsymbol{\theta}}(\theta)$, expressed in polar coordinates, to be conservative, i.e. be of the form $(\nabla\psi)(r,\theta)$ for some $\psi(r,\theta)$.

Reminder: Midterm I is on Wednesday, February 10.

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