Math 227 Midterm 1 Solutions

1. \( \mathbf{r}(t) = \left( e^t (\cos t - \sin t), e^t (\cos t + \sin t), e^t \right) \)
   
   \( \mathbf{v}(t) = e^t (\cos t - \sin t, \cos t + \sin t, 1) + e^t (-\sin t - \cos t, -\sin t + \cos t, 0) \)
   
   \( |\mathbf{v}(t)| = e^t \sqrt{(-2 \sin t)^2 + (2 \cos t)^2 + 1} \)
   
   \( \mathbf{T}(t) = \frac{1}{\sqrt{5}} (-2 \sin t, 2 \cos t, 1) \)
   
   \( \frac{d}{dt} \mathbf{T}(t) = \frac{1}{\sqrt{5}} (-2 \cos t, -2 \sin t, 0) \)
   
   \( \frac{d}{dt} \mathbf{T}(0) = \frac{1}{\sqrt{5}} (-2, 0, 0) = \kappa \hat{\mathbf{N}}(0) \frac{d}{dt} \)

   (a) The arc length is \( s = \int_0^\pi |\mathbf{v}| \, dt = \sqrt{5} \int_0^\pi e^t \, dt = \frac{5}{2} e^\pi - 1 \)

   (b) At time 0, \( \mathbf{T} = \frac{1}{\sqrt{5}} (0, 2, 1), \mathbf{N} = (-1, 0, 0), \kappa = \frac{2}{\sqrt{5}}, \mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{1}{\sqrt{5}} (0, -1, 2) \)

   (c) The osculating circle has radius \( \rho = 5/2 \), centre \( \mathbf{r}_c = \mathbf{r}(0) + \frac{1}{\rho} \hat{\mathbf{N}}(0) = (1, 1, 1) + \frac{2}{5} (-1, 0, 0) = (-3/2, 1, 1) \)

   and lies a plane spanned by \( \mathbf{T}(0) \) and \( \hat{\mathbf{N}}(0) \). It is given parametrically by
   
   \[ \mathbf{r}(t) = \mathbf{r}_c + \rho \cos t \mathbf{T}(0) + \rho \sin t \hat{\mathbf{N}}(0) = \left( \frac{1}{2} - \frac{2}{5} \sin t \right) \mathbf{i} + \left( 1 + \sqrt{5} \cos t \right) \mathbf{j} + \left( 1 + \frac{\sqrt{2}}{5} \cos t \right) \mathbf{k} \]

2. All of the streamlines of \( \vec{F} = z \mathbf{i} + z \mathbf{j} - (3x + y) \mathbf{k} \) obey

   \[ \frac{dx}{z} = \frac{du}{z} = -\frac{dz}{3x + y} \implies dy = dx \implies y = x + C \]

   For the field line to pass through \((1, 1, 2)\) we need \(1 = 1 + C\) and hence \(C = 0\). So \(x = y\) and

   \[ \frac{dx}{z} = -\frac{dz}{3x + y} \implies \frac{dx}{z} = -\frac{dz}{4x} \implies z \, dz = -4x \, dx \implies \frac{1}{2} z^2 = -2x^2 + C' \implies z^2 = -4x^2 + 2C' \]

   For the field line to pass through \((1, 1, 2)\) we need \(4 = -4 + 2C'\) and hence \(2C' = 8\). So the stream line is \( y = x, z = \sqrt{8 - 4x^2} \).

   **Warnings.** (a) When you use separation of variables to solve, for example, \( \frac{dy}{z} = -\frac{dz}{3x + y} \), you must first somehow eliminate all \(x\)’s and then move all \(y\)’s to the left hand side of the equation, and all \(z\)’s to the right hand side of the equation, before integrating. Integrating \( \int (3x + y) \, dy = 3xy + \frac{y^2}{2} + C \) is wrong, because \(x\) is not a constant. It is some (possibly unknown) function of \(y\) (and possibly \(z\)). In this example, \(x = y\).

   (b) An equation of the form \( f(x, y, z) = 0 \) is the equation of a surface in \( \mathbb{R}^3 \), not the equation of a curve in \( \mathbb{R}^3 \). To get a curve in \( \mathbb{R}^3 \) you need two equations in three unknowns, not one equation in three unknowns.

3. (a) The field is conservative only if

   \[ \frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}, \quad \frac{\partial F_z}{\partial x} = \frac{\partial F_x}{\partial z}, \quad \frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y} \]

   see over
That is,

\[
\frac{\partial}{\partial y} (2x \sin(\pi y) - e^z) = \frac{\partial}{\partial x} \left( ax^2 \cos(\pi y) - 3e^z \right) \iff 2\pi x \cos(\pi y) = 2ax \cos(\pi y)
\]

\[
\frac{\partial}{\partial z} (2x \sin(\pi y) - e^z) = - \frac{\partial}{\partial x} (x + by) e^z \iff -e^z = -e^z
\]

\[
\frac{\partial}{\partial x} (ax^2 \cos(\pi y) - 3e^z) = - \frac{\partial}{\partial y} (x + by) e^z \iff -3e^z = -be^z
\]

Hence only \( a = \pi, \ b = 3 \) works.

(b) When \( a = \pi, \ b = 3 \)

\[
\vec{F} = (2x \sin(\pi y) - e^z) \hat{i} + (\pi x^2 \cos(\pi y) - 3e^z) \hat{j} - (x + 3y) e^z \hat{k}
\]

\[
= \nabla (x^2 \sin(\pi y) - xe^z - 3ye^z + C)
\]

so \( \phi(x, y, z) = x^2 \sin(\pi y) - xe^z - 3ye^z + C \) for any constant \( C \).

4. Observe that \( \vec{G} = \vec{F} + 3ye^z \hat{k} \), with the \( \vec{F} \) of #3 evaluated with \( a = \pi, \ b = 3 \). Hence

\[
\int_C \vec{G} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} + \int_C 3ye^z \hat{k} \cdot d\vec{r} = \phi(1,1,\ln 2) - \phi(0,0,0) + \int_C 3ye^z \hat{k} \cdot d\vec{r} = -8 + \int_C 3ye^z \hat{k} \cdot d\vec{r}
\]

To evaluate the remaining integral, parametrize the curve by \( \vec{r}(t) = t\hat{i} + t\hat{j} + \ln(1+t)\hat{k} \). Then \( \vec{r}'(t) = \hat{i} + \hat{j} + \frac{1}{1+t} \hat{k} \) and \( 3ye^z = 3t(1+t)\hat{k} \) so that \( 3ye^z \hat{k} \cdot d\vec{r} = 3t \, dt \). Subbing in

\[
\int_C \vec{G} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} = -8 + \int_0^1 3t \, dt = -8 + \frac{3}{2} = \frac{-13}{2}
\]

see over