Hyperbolic Trig Functions

The hyperbolic trig functions are defined by
\[
\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}
\]
\[
\csc h x = \frac{2}{e^x - e^{-x}}, \quad \sech x = \frac{2}{e^x + e^{-x}}, \quad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}
\]

As their names suggest, these functions are very closely related to the trig functions. This relationship may be seen from the formulae
\[
\sin x = \frac{e^{ix} - e^{-ix}}{2i}, \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \tan x = -i \frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}}
\]
\[
\csc x = \frac{2i}{e^{ix} - e^{-ix}}, \quad \sec x = \frac{2}{e^{ix} + e^{-ix}}, \quad \cot x = i \frac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}}
\]

(If you are not familiar with these formulae, see the handout entitled “Complex Numbers and Exponentials”.) In particular
\[
\sinh x = i \sin(-ix) \quad \cosh x = \cos(-ix)
\]

Consequently, the differentiation formulae for hyperbolic trig functions are almost identical to those for trig functions:
\[
\frac{d}{dx} \sinh x = \cosh x \quad \frac{d}{dx} \cosh x = \sinh x
\]

They differ only by some sign changes. Similarly, for each trig identity there is a corresponding hyperbolic trig identity, which is also identical up to sign changes:
\[
\cosh^2 x - \sinh^2 x = \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = 1
\]
\[
\sinh 2x = \frac{e^{2x} - e^{-2x}}{2} = 2 \sinh x \cosh x
\]
\[
\cosh 2x = \frac{e^{2x} + e^{-2x}}{2} = 2 \cosh^2 x - 1
\]

Example. Find \( \int \sqrt{a^2 + x^2} \, dx \).

Solution. The standard technique for integrating \( \int \sqrt{a^2 - x^2} \, dx \) is to substitute \( x = a \sin t \) and exploit the trig identity \( 1 - \sin^2 t = \cos^2 t \) to eliminate the square root. The analog here is to substitute \( x = a \sinh t \) (\( \sinh t \) is a strictly increasing function, so the change of variables is legitimate) and exploit \( 1 + \sinh^2 t = \cosh^2 t \), which we do. Since \( dx = a \cosh t \, dt \),
\[
\int \sqrt{a^2 + x^2} \, dx = \int \sqrt{a^2 + a^2 \sinh^2 t} \, a \cosh t \, dt = \int \sqrt{a^2 \cosh^2 t} \, a \cosh t \, dt
\]
\[
= \int a^2 \cosh^2 t \, dt
\]
Note that \( \cosh t > 0 \) for all \( t \), so we have correctly taken the positive square root. The standard technique for integrating \( \int \cos^2 t \, dt \) exploits the trig identity \( \cos^2 t = \frac{1+\cos 2t}{2} \). To integrate \( \int a^2 \cosh^2 t \, dt \) we use \( \cosh^2 t = \frac{1+\cosh 2t}{2} \).

\[
\int a^2 \cosh^2 t \, dt = a^2 \int \frac{1+\cosh 2t}{2} \, dt = \frac{a^2}{2} \left[ t + \frac{1}{2} \sinh 2t \right] + C
\]

To get back to the original variable \( x \), sub in \( t = \sinh^{-1} \frac{x}{a} \) and use the identities \( \sinh 2t = 2 \sinh t \cosh t \) and \( \cosh t = \sqrt{\sinh^2 t + 1} \).

\[
\frac{a^2}{2} \left[ t + \frac{1}{2} \sinh 2t \right] = \frac{a^2}{2} \left[ \sinh^{-1} \frac{x}{a} + \frac{1}{2} x \sqrt{x^2 + a^2} \right]
\]

All together

\[
\int \sqrt{a^2 + x^2} \, dx = \frac{a^2}{2} \sinh^{-1} \frac{x}{a} + \frac{1}{2} x \sqrt{x^2 + a^2} + C
\]

In addition, the inverse hyperbolic trig function \( \sinh^{-1} x \) can be explicitly expressed in terms of \( \ln \)'s. By definition, \( y = \sinh^{-1} x \) is the unique solution of \( \sinh y = x \), or

\[
\frac{e^y - e^{-y}}{2} = x \quad \Rightarrow \quad e^y - e^{-y} = 2x \quad \Rightarrow \quad e^{2y} - 1 = 2xe^y \quad \Rightarrow \quad e^{2y} - 2xe^y - 1 = 0
\]

Think of this as a quadratic equation in \( e^y \), with \( x \) just being a given constant. The solution, by the high school formula for the solution of a quadratic equation is

\[
e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}
\]

As \( y \) is a real number, \( e^y \) must be a positive number and we must reject the negative sign. Thus

\[
y = \sinh^{-1} x = \ln \left( x + \sqrt{x^2 + 1} \right)
\]

and we may rewrite the above integral as

\[
\int \sqrt{a^2 + x^2} \, dx = \frac{a^2}{2} \ln \left( x + \sqrt{x^2 + a^2} \right) + \frac{1}{2} x \sqrt{x^2 + a^2} + C'
\]

(with the new constant \( C' = C - \frac{a^2}{2} \ln a \)).