

Flux Integral Example

Problem: Evaluate $\iint_S \vec{F} \cdot \hat{n} \, dS$ where $\vec{F} = x^4 \hat{i} + 2y^2 \hat{j} + z \hat{k}$, S is the half of the surface $\frac{1}{4}x^2 + \frac{1}{9}y^2 + z^2 = 1$ with $z \geq 0$ and \hat{n} is the upward unit normal.

Solution 1. Parametrize the half-ellipsoid

$$x(\theta, \phi) = 2 \cos \theta \sin \phi \quad y(\theta, \phi) = 3 \sin \theta \sin \phi \quad z(\theta, \phi) = \cos \phi$$

with (θ, ϕ) running over $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \pi/2$. Then

$$\left(\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta} \right) = (-2 \sin \theta \sin \phi, 3 \cos \theta \sin \phi, 0)$$

$$\left(\frac{\partial x}{\partial \phi}, \frac{\partial y}{\partial \phi}, \frac{\partial z}{\partial \phi} \right) = (2 \cos \theta \cos \phi, 3 \sin \theta \cos \phi, -\sin \phi)$$

$$\hat{n} dS = -\left(\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta} \right) \times \left(\frac{\partial x}{\partial \phi}, \frac{\partial y}{\partial \phi}, \frac{\partial z}{\partial \phi} \right) d\theta d\phi$$

$$= -(-3 \cos \theta \sin^2 \phi, -2 \sin \theta \sin^2 \phi, -6 \sin \phi \cos \phi) d\theta d\phi$$

$$\vec{F} = 2^4 \cos^4 \theta \sin^4 \phi \hat{i} + 2 \times 3^2 \sin^2 \theta \sin^2 \phi \hat{j} + \cos \phi \hat{k}$$

$$\vec{F} \cdot \hat{n} \, dS = \left[3 \times 2^4 \cos^5 \theta \sin^6 \phi + 2 \times 3^2 \times 2 \sin^3 \theta \sin^4 \phi + 6 \sin \phi \cos^2 \phi \right] d\theta d\phi$$

The extra minus sign in $\hat{n} dS$ was put there to make the z component of \hat{n} positive. Since $\int_0^{2\pi} \cos^m \theta \, d\theta = \int_0^{2\pi} \sin^m \theta \, d\theta = 0$ for all odd integers m

$$\int_S \vec{F} \cdot \hat{n} \, dS = \int_0^{\pi/2} d\phi \int_0^{2\pi} d\theta \, 6 \sin \phi \cos^2 \phi = 12\pi \int_0^{\pi/2} d\phi \sin \phi \cos^2 \phi = 12\pi \left[-\frac{1}{3} \cos^3 \phi \right]_0^{\pi/2} = \boxed{4\pi}$$

Solution 2. Parametrize the half-ellipsoid

$$x(r, \theta) = 2r \cos \theta \quad y(r, \theta) = 3r \sin \theta \quad z(r, \theta) = \sqrt{1 - r^2}$$

with (r, θ) running over $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 1$. Then

$$\left(\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta} \right) = (-2r \sin \theta, 3r \cos \theta, 0)$$

$$\left(\frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}, \frac{\partial z}{\partial r} \right) = \left(2 \cos \theta, 3 \sin \theta, -\frac{r}{\sqrt{1-r^2}} \right)$$

$$\hat{n} dS = -\left(\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta} \right) \times \left(\frac{\partial x}{\partial r}, \frac{\partial y}{\partial r}, \frac{\partial z}{\partial r} \right) dr d\theta$$

$$= -\left(-\frac{3r^2 \cos \theta}{\sqrt{1-r^2}}, -\frac{2r^2 \sin \theta}{\sqrt{1-r^2}}, -6r \right) dr d\theta$$

$$\vec{F} = 2^4 r^4 \cos^4 \theta \hat{i} + 2 \times 3^2 r^2 \sin^2 \theta \hat{j} + \sqrt{1 - r^2} \hat{k}$$

$$\vec{F} \cdot \hat{n} \, dS = \left[3 \times 2^4 \frac{r^6}{\sqrt{1-r^2}} \cos^5 \theta + 2^2 3^2 \frac{r^4}{\sqrt{1-r^2}} \sin^3 \theta + 6r \sqrt{1 - r^2} \right] dr d\theta$$

The extra minus sign in $\hat{n} dS$ was put there to make the z component of \hat{n} positive. Since $\int_0^{2\pi} \cos^m \theta \, d\theta = \int_0^{2\pi} \sin^m \theta \, d\theta = 0$ for all odd integers m

$$\int_S \vec{F} \cdot \hat{n} \, dS = \int_0^1 dr \int_0^{2\pi} d\theta \, 6r \sqrt{1 - r^2} = 12\pi \int_0^1 dr \, r \sqrt{1 - r^2} = 12\pi \left[-\frac{1}{3} (1 - r^2)^{3/2} \right]_0^1 = \boxed{4\pi}$$

Solution 3. The surface is of the form $g(x, y, z) = 0$ with $g(x, y, z) = \frac{1}{4}x^2 + \frac{1}{9}y^2 + z^2 - 1$. Hence

$$\begin{aligned}\hat{\mathbf{n}}dS &= \frac{\vec{\nabla}g}{\vec{\nabla}g \cdot \hat{\mathbf{k}}} dx dy = \frac{\frac{x}{2}\hat{\mathbf{i}} + \frac{2y}{9}\hat{\mathbf{j}} + 2z\hat{\mathbf{k}}}{2z} dx dy = \left(\frac{x}{z}\hat{\mathbf{i}} + \frac{y}{9z}\hat{\mathbf{j}} + \hat{\mathbf{k}}\right) dx dy \\ \Rightarrow \vec{F} \cdot \hat{\mathbf{n}} dS &= \left(\frac{x^5}{z} + \frac{2y^3}{9z} + z\right) dx dy\end{aligned}$$

On S , $z = z(x, y) = \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}$ and $\frac{x^2}{4} + \frac{y^2}{9} \leq 1$, so

$$\int_S \vec{F} \cdot \hat{\mathbf{n}} dS = \iint_{\frac{x^2}{4} + \frac{y^2}{9} \leq 1} \left(\frac{x^5}{z(x,y)} + \frac{2y^3}{9z(x,y)} + z(x,y)\right) dx dy = \iint_{\frac{x^2}{4} + \frac{y^2}{9} \leq 1} z(x,y) dx dy$$

since both $\frac{x^5}{z(x,y)}$ and $\frac{2y^3}{9z(x,y)}$ are odd under $x \rightarrow -x$, $y \rightarrow -y$ and the domain of integration is even under $x \rightarrow -x$, $y \rightarrow -y$. So

$$\int_S \vec{F} \cdot \hat{\mathbf{n}} dS = \iint_{\frac{x^2}{4} + \frac{y^2}{9} \leq 1} \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}} dx dy$$

To evaluate this integral, make the change of variables

$$x = 2r \cos \theta \quad y = 3r \sin \theta$$

Then the domain of integration, $\frac{x^2}{4} + \frac{y^2}{9} \leq 1$ becomes $r^2 \leq 1$, the integrand $\sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}$ becomes $\sqrt{1 - r^2}$ and $dx dy$ becomes

$$\left| \det \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} \right| dr d\theta = \left| \det \begin{bmatrix} 2 \cos \theta & -2r \sin \theta \\ 3 \sin \theta & 3r \cos \theta \end{bmatrix} \right| dr d\theta = 6r dr d\theta$$

So

$$\int_S \vec{F} \cdot \hat{\mathbf{n}} dS = \int_0^1 dr \int_0^{2\pi} d\theta 6r \sqrt{1 - r^2} = 12\pi \int_0^1 dr r \sqrt{1 - r^2} = 12\pi \left[-\frac{1}{3}(1 - r^2)^{3/2} \right]_0^1 = \boxed{4\pi}$$

Solution 4. The surface is of the form $z = f(x, y)$ with $f(x, y) = \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}$. Hence

$$\begin{aligned}\hat{\mathbf{n}}dS &= \left[-\frac{\partial f}{\partial x}\hat{\mathbf{i}} - \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \hat{\mathbf{k}} \right] dx dy = \left[\frac{\frac{x}{4}\hat{\mathbf{i}} + \frac{y}{9}\hat{\mathbf{j}}}{\sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}} + \hat{\mathbf{k}} \right] dx dy \\ \Rightarrow \vec{F} \cdot \hat{\mathbf{n}} dS &= \left[\frac{\frac{x^5}{4} + \frac{2y^3}{9}}{\sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}}} + \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}} \right] dx dy\end{aligned}$$

As in Solution 3, by oddness,

$$\int_S \vec{F} \cdot \hat{\mathbf{n}} dS = \iint_{\frac{x^2}{4} + \frac{y^2}{9} \leq 1} \left[\frac{\frac{x^5}{4} + \frac{2y^3}{9}}{\sqrt{\dots}} + \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}} \right] dx dy = \iint_{\frac{x^2}{4} + \frac{y^2}{9} \leq 1} \sqrt{1 - \frac{x^2}{4} - \frac{y^2}{9}} dx dy$$

Now continue as in Solution 3.