Flux Formulae

Parametrized Surfaces. If the surface is

$$x = x(\theta, \phi) \qquad y = y(\theta, \phi) \qquad z = z(\theta, \phi)$$

then, setting,

$$\vec{T}_\theta = \big(\tfrac{\partial x}{\partial \theta}, \tfrac{\partial y}{\partial \theta}, \tfrac{\partial z}{\partial \theta}\big) \qquad \vec{T}_\phi = \big(\tfrac{\partial x}{\partial \phi}, \tfrac{\partial y}{\partial \phi}, \tfrac{\partial z}{\partial \phi}\big)$$

we have

$$\begin{aligned} dS &= \|\vec{T}_\theta \times \vec{T}_\phi\|\ d\theta\, d\phi \\ \hat{n} &= \pm \frac{\vec{T}_\theta \times \vec{T}_\phi}{\|\vec{T}_\theta \times \vec{T}_\phi\|} \\ \hat{n} dS &= \pm\, \vec{T}_\theta \times \vec{T}_\phi\ d\theta\, d\phi \end{aligned}$$

Level Surfaces. If the surface is $g(x, y, z) = 0$, then

$$\begin{aligned} \hat{\mathbf{n}} &= \pm \frac{\vec{\nabla} g}{\|\vec{\nabla} g\|} \\ dS &= \left\| \frac{\vec{\nabla} g}{\vec{\nabla} g \cdot \hat{\mathbf{k}}} \right\| dx\, dy = \left\| \frac{\vec{\nabla} g}{\vec{\nabla} g \cdot \hat{\boldsymbol{\imath}}} \right\| dy\, dz = \left\| \frac{\vec{\nabla} g}{\vec{\nabla} g \cdot \hat{\boldsymbol{\jmath}}} \right\| dx\, dz \\ \hat{\mathbf{n}} dS &= \pm \frac{\vec{\nabla} g}{\vec{\nabla} g \cdot \hat{\mathbf{k}}} dx\, dy \quad = \pm \frac{\vec{\nabla} g}{\vec{\nabla} g \cdot \hat{\boldsymbol{\imath}}} dy\, dz \quad = \pm \frac{\vec{\nabla} g}{\vec{\nabla} g \cdot \hat{\boldsymbol{\jmath}}} dx\, dz \end{aligned}$$

Graphs. If the surface is $z = f(x, y)$ or $x = g(y, z)$ or $y = h(x, z)$ then

$$dS = \sqrt{1 + \big(\tfrac{\partial f}{\partial x}\big)^2 + \big(\tfrac{\partial f}{\partial y}\big)^2}\ dx\, dy \qquad dS = \sqrt{1 + \big(\tfrac{\partial g}{\partial y}\big)^2 + \big(\tfrac{\partial g}{\partial z}\big)^2}\ dy\, dz \qquad dS = \sqrt{1 + \big(\tfrac{\partial h}{\partial x}\big)^2 + \big(\tfrac{\partial h}{\partial z}\big)^2}\ dx\, dz$$

$$\hat{\mathbf{n}} = \pm \frac{-\frac{\partial f}{\partial x}\hat{\boldsymbol{\imath}} - \frac{\partial f}{\partial y}\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}}{\sqrt{1 + \big(\tfrac{\partial f}{\partial x}\big)^2 + \big(\tfrac{\partial f}{\partial y}\big)^2}} \qquad \hat{\mathbf{n}} = \pm \frac{\hat{\boldsymbol{\imath}} - \frac{\partial g}{\partial y}\hat{\boldsymbol{\jmath}} - \frac{\partial g}{\partial z}\hat{\mathbf{k}}}{\sqrt{1 + \big(\tfrac{\partial g}{\partial y}\big)^2 + \big(\tfrac{\partial g}{\partial z}\big)^2}} \qquad \hat{\mathbf{n}} = \pm \frac{-\frac{\partial h}{\partial x}\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - \frac{\partial h}{\partial z}\hat{\mathbf{k}}}{\sqrt{1 + \big(\tfrac{\partial h}{\partial x}\big)^2 + \big(\tfrac{\partial h}{\partial z}\big)^2}}$$

$$\hat{\mathbf{n}} dS = \pm\Big[-\tfrac{\partial f}{\partial x}\hat{\boldsymbol{\imath}} - \tfrac{\partial f}{\partial y}\hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}\Big] dx\, dy \qquad \hat{\mathbf{n}} dS = \pm\Big[\hat{\boldsymbol{\imath}} - \tfrac{\partial g}{\partial y}\hat{\boldsymbol{\jmath}} - \tfrac{\partial g}{\partial z}\hat{\mathbf{k}}\Big] dy\, dz \qquad \hat{\mathbf{n}} dS = \pm\Big[-\tfrac{\partial h}{\partial x}\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - \tfrac{\partial h}{\partial z}\hat{\mathbf{k}}\Big] dx\, dz$$