

## Faraday's law

Faraday's Law is the following. Let  $S$  be an oriented surface with boundary  $C$ . Let  $\vec{E}$  and  $\vec{H}$  be the (time dependent) electric and magnetic fields and define

$$\oint_C \vec{E} \cdot d\vec{r} = \text{voltage around } C \quad \iint_S \vec{H} \cdot \hat{\mathbf{n}} dS = \text{magnetic flux through } S$$

Then the voltage around  $C$  is the negative of the rate of change of the magnetic flux through  $S$ . As an equation, Faraday's Law is

$$\oint_C \vec{E} \cdot d\vec{r} = -\frac{\partial}{\partial t} \iint_S \vec{H} \cdot \hat{\mathbf{n}} dS$$

By Stokes' Theorem

$$\oint_C \vec{E} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{E}) \cdot \hat{\mathbf{n}} dS$$

so

$$\iint_S \left( \vec{\nabla} \times \vec{E} + \frac{\partial \vec{H}}{\partial t} \right) \cdot \hat{\mathbf{n}} dS = 0$$

This is true for all surfaces  $S$ . So the integrand, assuming that it is continuous, must be zero. To see this, let  $\vec{G} = \left( \vec{\nabla} \times \vec{E} + \frac{\partial \vec{H}}{\partial t} \right)$ . Suppose that  $\vec{G}(\vec{x}_0) \neq 0$ . Pick a unit vector  $\hat{\mathbf{n}}$  in the direction of  $\vec{G}(\vec{x}_0)$ . Let  $S$  be a very small flat disk centered on  $\vec{x}_0$  with normal  $\hat{\mathbf{n}}$  (the vector we picked). Then  $\vec{G}(\vec{x}_0) \cdot \hat{\mathbf{n}} > 0$  and, by continuity,  $\vec{G}(\vec{x}) \cdot \hat{\mathbf{n}} > 0$  for all  $\vec{x}$  on  $S$ , if we have picked  $S$  small enough. Then  $\iint_S \left( \vec{\nabla} \times \vec{E} + \frac{\partial \vec{H}}{\partial t} \right) \cdot \hat{\mathbf{n}} dS > 0$ , which is a contradiction. So we conclude that

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{H}}{\partial t} = 0$$

This is one of Maxwell's equations.