Divergence Theorem and Variations

**Theorem.** If \( V \) is a solid with surface \( \partial V \)

\[
\int\int_{\partial V} \mathbf{F} \cdot \mathbf{n} \, dS = \int\int\int_V \nabla \cdot \mathbf{F} \, dV \\
\int\int_{\partial V} f \mathbf{n} \, dS = \int\int\int_V \nabla f \, dV \\
\int\int_{\partial V} \mathbf{n} \times \mathbf{F} \, dS = \int\int\int_V \nabla \times \mathbf{F} \, dV
\]

where \( \mathbf{n} \) is the outward unit normal of \( \partial V \).

**Memory Aid.** All three formulae can be combined into

\[
\int\int_{\partial V} \mathbf{n} \ast \tilde{\mathbf{F}} \, dS = \int\int\int_V \nabla \ast \tilde{\mathbf{F}} \, dV
\]

where \( \ast \) can be either \( \cdot \), \( \times \) or nothing. When \( \ast = \cdot \) or \( \ast = \times \), then \( \tilde{\mathbf{F}} = \mathbf{F} \). When \( \ast \) is nothing, \( \tilde{\mathbf{F}} = f \).

**Proof:** The first formula is the divergence theorem and was proven in class.

To prove the second formula, assuming the first, apply the first with \( \mathbf{F} = f \mathbf{a} \), where \( \mathbf{a} \) is any constant vector.

\[
\int\int_{\partial V} f \mathbf{a} \cdot \mathbf{n} \, dS = \int\int\int_V \nabla \cdot (f \mathbf{a}) \, dV \\
= \int\int\int_V [(\nabla f) \cdot \mathbf{a} + f \nabla \cdot \mathbf{a}] \, dV \\
= \int\int\int_V (\nabla f) \cdot \mathbf{a} \, dV
\]

To get the second line, we used vector identity \# 8. To get the third line, we just used that \( \mathbf{a} \) is a constant, so that it is annihilated by all derivatives. Since \( \mathbf{a} \) is a constant, we can factor it out of both integrals, so

\[
\mathbf{a} \cdot \int\int_{\partial V} f \mathbf{n} \, dS = \mathbf{a} \cdot \int\int\int_V \nabla f \, dV \\
\Rightarrow \mathbf{a} \cdot \left\{ \int\int_{\partial V} f \mathbf{n} \, dS - \int\int\int_V \nabla f \, dV \right\} = 0
\]
In particular, choosing \( \mathbf{a} = \hat{i}, \hat{j}, \) and \( \hat{k}, \) we see that all three components of the vector 
\[
\iint_{\partial V} \mathbf{f} \hat{n} dS - \iiint_{V} \nabla f \, dV \quad \text{are zero. So}
\]
\[
\iint_{\partial V} \mathbf{f} \hat{n} dS - \iiint_{V} \nabla f \, dV = 0
\]

which is what we wanted show.

To prove the third formula, assuming the first, apply the first with \( \mathbf{F} \) replaced by \( \mathbf{F} \times \mathbf{a}, \) where \( \mathbf{a} \) is any constant vector.

\[
\iint_{\partial V} \mathbf{F} \times \mathbf{a} \cdot \hat{n} \, dS = \iint_{\partial V} \nabla \cdot (\mathbf{F} \times \mathbf{a}) \, dV
\]
\[
= \iint_{\partial V} \left[ (\nabla \times \mathbf{F}) \cdot \mathbf{a} - \mathbf{F} \cdot \nabla \times \mathbf{a} \right] \, dV
\]
\[
= \iint_{\partial V} (\nabla \times \mathbf{F}) \cdot \mathbf{a} \, dV
\]

To get the second line, we used vector identity \# 9. To get the third line, we just used that \( \mathbf{a} \) is a constant, so that it is annihilated by all derivatives. For all vectors

\[
\mathbf{F} \times \mathbf{a} \cdot \hat{n} = \hat{n} \cdot \mathbf{F} \times \mathbf{a} = \hat{n} \times \mathbf{F} \cdot \mathbf{a}
\]

so

\[
\mathbf{a} \cdot \iint_{\partial V} \hat{n} \times \mathbf{F} \, dS = \mathbf{a} \cdot \iint_{V} \nabla \times \mathbf{F} \, dV
\]
\[
\Rightarrow \mathbf{a} \cdot \left\{ \iint_{\partial V} \hat{n} \times \mathbf{F} \, dS - \iint_{V} \nabla \times \mathbf{F} \, dV \right\} = 0
\]

In particular, choosing \( \mathbf{a} = \hat{i}, \hat{j}, \) and \( \hat{k}, \) we see that all three components of the vector 
\[
\iint_{\partial V} \hat{n} \times \mathbf{F} \, dS - \iiint_{V} \nabla \times \mathbf{F} \, dV \quad \text{are zero. So}
\]
\[
\iint_{\partial V} \hat{n} \times \mathbf{F} \, dS - \iiint_{V} \nabla \times \mathbf{F} \, dV = 0
\]

which is what we wanted show. \( \blacksquare \)