A Compendium of Curve Formulae

In the following \( \mathbf{r}(t) = (x(t), y(t), z(t)) \) is a parametrization of a curve. The vectors \( \mathbf{T}(t), \mathbf{N}(t), \) and \( \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) \) are the unit tangent, normal and binormal vectors, respectively, at \( \mathbf{r}(t) \). The tangent vector points in the direction of travel (i.e. direction of increasing \( t \)) and the normal vector points toward the centre of curvature. The arc length from time 0 to time \( t \) is denoted \( s(t) \). Then

- the velocity \( \mathbf{v}(t) = \frac{d\mathbf{r}}{dt}(t) = \frac{ds}{dt}(t) \mathbf{T}(t) \)
- the acceleration \( \mathbf{a}(t) = \frac{d^2\mathbf{r}}{dt^2}(t) = \frac{d^2s}{dt^2}(t) \mathbf{T}(t) + \kappa(t) \left( \frac{ds}{dt}(t) \right)^2 \mathbf{N}(t) \)
- the speed \( \frac{ds}{dt}(t) = |\mathbf{v}(t)| = |d\mathbf{r}/dt| \)
- the arc length \( s(t) = \int_0^1 \frac{ds}{dt}(\tau) \, d\tau = \int_0^1 \sqrt{x'(\tau)^2 + y'(\tau)^2 + z'(\tau)^2} \, d\tau \)
- the curvature \( \kappa(t) = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{(\frac{ds}{dt}(t))^3} \)
- the radius of curvature \( \rho(t) = \frac{1}{\kappa(t)} \)
- the centre of curvature is \( \mathbf{r}(t) + \rho(t) \mathbf{N}(t) \)
- the torsion \( \tau(t) = \frac{(\mathbf{v}(t) \times \mathbf{a}(t)) \cdot \frac{d\mathbf{a}}{dt}(t)}{|\mathbf{v}(t) \times \mathbf{a}(t)|^2} \)
- the binormal \( \hat{\mathbf{B}}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{\mathbf{v}(t) \times \mathbf{a}(t)}{|\mathbf{v}(t) \times \mathbf{a}(t)|} \)

Under arc parametrization (i.e. if \( t = s \)) we have \( \hat{\mathbf{T}}(s) = \frac{d\mathbf{r}}{ds}(s) \) and the Frenet-Serret formulae

\[
\frac{d\mathbf{T}}{ds}(s) = \kappa(s) \mathbf{N}(s) \\
\frac{d\mathbf{N}}{ds}(s) = \tau(s) \mathbf{B}(s) - \kappa(s) \hat{\mathbf{T}}(s) \\
\frac{d\mathbf{B}}{ds}(s) = -\tau(s) \mathbf{N}(s)
\]

When the curve lies in the \( x-y \) plane

\[
\kappa(t) = \frac{|\frac{dx}{dt}(t) \frac{dy}{dt}(t) - \frac{dy}{dt}(t) \frac{dx}{dt}(t)|}{\left[ (\frac{dx}{dt}(t))^2 + (\frac{dy}{dt}(t))^2 \right]^{3/2}}
\]

When the curve lies in the \( x-y \) plane and the parameter is \( x \) (so that \( y \) is given as a function \( y(x) \) of \( x \))

\[
\kappa(x) = \frac{|\frac{d^2y}{dx^2}(x)|}{\left[ 1 + \left( \frac{dy}{dx}(x) \right)^2 \right]^{3/2}}
\]