Skateboarding in a Culvert

First, consider a bead of mass $m$ that is sliding, without friction, on a stiff wire. It is subject to two forces. The gravitational force is $-mg\hat{j}$. By definition, absence of friction means that the wire is incapable of applying any force in the direction tangential to the wire on the bead. But, because it is stiff, the wire never changes shape and instead applies just the right amount of force, in the direction normal to the wire, that is needed to keep the bead on the wire, without bending the wire. Call this normal force $W\hat{N}$.

By Newton’s law

$$ma = -mg\hat{j} + W\hat{N}$$

To extract the tangential component of Newton’s law, we dot it with $\hat{v}$:

$$mv \cdot \frac{dv}{dt} = -mg\hat{j} \cdot v + W\hat{N} \cdot v$$

$$\frac{1}{2}m\frac{d}{dt}|v|^2 = -mg\frac{dv}{dt}$$

since $\hat{j} \cdot v$ is just the $y$ component of $v$ and since $\hat{N}$ and $v = |v|\hat{T}$ are perpendicular. Moving everything to the left hand side of the equation gives

$$\frac{d}{dt}\left(\frac{1}{2}m|v|^2 + mgy\right) = 0$$

and we conclude that

$$E = \frac{1}{2}m|v|^2 + mgy$$

(E)

is a constant, independent of time. This is, of course, conservation of energy. To extract the normal component of Newton’s law, we dot it with $\hat{N}$:

$$ma \cdot \hat{N} = -mg\hat{j} \cdot \hat{N} + W$$

Since

$$a(t) = \frac{d^2s}{dt^2}\hat{T} + \kappa\left(\frac{ds}{dt}\right)^2\hat{N}$$

and $\hat{T}$ and $\hat{N}$ are perpendicular, this gives

$$W = m\kappa|v|^2 + mg\hat{j} \cdot \hat{N} = 2\kappa(E - mgy) + mg\hat{j} \cdot \hat{N}$$

(W)
So far, equations (E,W) apply to any stiff frictionless “wire”. We now specialize to the special case of a skateboarder inside a circular culvert of radius \( a \). Let’s put the bottom of the circle at the origin \((0,0)\), so that the centre of the circle is at \((0,a)\).

In this case the curvature is \( \kappa = \frac{1}{a} \) and \( \hat{\mathbf{j}} \cdot \hat{\mathbf{N}} = \cos \phi = \frac{a-y}{a} \) so (E) and (W) simplify to

\[
|v| = \sqrt{\frac{2}{m}(E - mgy)} = \sqrt{2g(\frac{E}{mg} - y)} \quad \text{ (E)}
\]

\[
W = \frac{2}{a}(E - mgy) + \frac{ma}{a}(a-y) = \frac{3ma}{a} \left( \frac{2}{3mg}E + \frac{a}{3} - y \right) \quad \text{ (W)}
\]

Imagine now that you start at the bottom of the culvert, that is at \( y = 0 \), with energy \( E > 0 \). As time progresses, \( y \) increases and consequently \( |v| \) and \( W \) both decrease. This continues until one of the following three things happen

(i) \( |v| \) hits 0, in which case you stop rising and start descending. The speed \( |v| \) is zero when \( y = y_S = \frac{E}{mg} \) (The subscript “S” stands for “stop”.)

(ii) \( W \) hits zero. When you get higher than this, \( W \) becomes negative and the culvert would have to pull on you to keep your feet on the culvert. As the culvert can only push on you, you become airborne. The normal force \( W \) is zero when \( y = y_A = \frac{2}{3}E + \frac{a}{3} \).

(The subscript “A” stands for “airborne”.)

(iii) \( y \) hits \( 2a \). This is the summit of the culvert. You descend on the other side.

Which case actually happens is determined by the relative sizes of \( y_S, y_A \) and \( 2a \).

- Comparing \( y_S = \frac{2}{3}E + \frac{a}{3} \) and \( y_A = \frac{2}{3}E + \frac{a}{3} \), we see that \( y_S \leq y_A \iff \frac{E}{mg} \leq a \).
- Comparing \( y_A = \frac{2}{3}E + \frac{a}{3} \) and \( 2a = \frac{5}{3}a + \frac{a}{3} \), we see that \( y_A \leq 2a \iff \frac{E}{mg} \leq \frac{5}{2}a \).
- Comparing \( y_A = \frac{2}{3}E + \frac{a}{3} \) and \( a = \frac{2}{3}a + \frac{a}{3} \), we see that \( y_A \leq a \iff \frac{E}{mg} \leq a \).

So the conclusions are:

- **If** \( 0 \leq \frac{E}{mg} \leq a \) **then** \( 0 \leq y_S \leq y_A \leq a \). In this case you just oscillate between heights 0 and \( y_S \leq a \) in the bottom half of the culvert.

- **If** \( a \leq \frac{E}{mg} \leq \frac{5}{2}a \) **then** \( a \leq y_A \leq y_S, 2a \). In this case you make it more than half way to the top. But you become airborne at \( y = y_A \) which is somewhere between the half way mark \( y = a \) and the top \( y = 2a \). At this point our model breaks down because you are no longer in contact with the culvert. You just free fall in a parabolic arc until you crash back into the culvert.

- **If** \( \frac{5}{2}a < \frac{E}{mg} \) **then** \( 2a < y_A < y_S \). In this case you successfully go all the way around the culvert, looping the loop.