

Circulation around a small circle

Let C_ε be the circle which

- is centered on $\vec{\mathbf{r}}_0$
- has radius ε
- lies in the plane through $\vec{\mathbf{r}}_0$ perpendicular to $\hat{\mathbf{n}}$
- is oriented in the standard way with respect to $\hat{\mathbf{n}}$. Imagine standing on the circle with your feet on the plane through $\vec{\mathbf{r}}_0$ perpendicular to $\hat{\mathbf{n}}$, with the vector from your feet to your head in the same direction as $\hat{\mathbf{n}}$ and with your left arm point towards $\vec{\mathbf{r}}_0$. Then you are facing in the positive direction for C_ε .

We shall show that

$$\oint_{C_\varepsilon} \vec{v}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}} = \pi\varepsilon^2 \vec{\nabla} \times v(\vec{\mathbf{r}}_0) \cdot \hat{\mathbf{n}} + O(\varepsilon^3)$$

To do so, pick any three vectors $\hat{\mathbf{i}}', \hat{\mathbf{j}}', \hat{\mathbf{k}}'$ such that

- $\hat{\mathbf{k}}' = \hat{\mathbf{n}}$
- $\hat{\mathbf{i}}' \perp \hat{\mathbf{n}}, \hat{\mathbf{j}}' \perp \hat{\mathbf{n}}$
- $\hat{\mathbf{i}}' \times \hat{\mathbf{j}}' = \hat{\mathbf{k}}'$

Then

$$\vec{\mathbf{r}}(t) = \vec{\mathbf{r}}_0 + \varepsilon \cos t \hat{\mathbf{i}}' + \varepsilon \sin t \hat{\mathbf{j}}'$$

is a parametrization of C_ε . Note in particular that, for all t , $\vec{\mathbf{r}}(t)$ lies in the plane through $\vec{\mathbf{r}}_0$ perpendicular to $\hat{\mathbf{n}}$ and $\|\vec{\mathbf{r}}(t) - \vec{\mathbf{r}}_0\| = \varepsilon$. So

$$\begin{aligned} \oint_{C_\varepsilon} \vec{v}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}} &= \int_0^{2\pi} \vec{v}(\vec{\mathbf{r}}_0 + \varepsilon \cos t \hat{\mathbf{i}}' + \varepsilon \sin t \hat{\mathbf{j}}') \cdot (-\varepsilon \sin t \hat{\mathbf{i}}' + \varepsilon \cos t \hat{\mathbf{j}}') dt \\ &= \varepsilon \int_0^{2\pi} \left[-\sin t \hat{\mathbf{i}}' \cdot \vec{v}(\vec{\mathbf{r}}_0 + \varepsilon \cos t \hat{\mathbf{i}}' + \varepsilon \sin t \hat{\mathbf{j}}') + \cos t \hat{\mathbf{j}}' \cdot \vec{v}(\vec{\mathbf{r}}_0 + \varepsilon \cos t \hat{\mathbf{i}}' + \varepsilon \sin t \hat{\mathbf{j}}') \right] dt \end{aligned}$$

Denote $f(\vec{\mathbf{r}}) = \hat{\mathbf{i}}' \cdot \vec{v}(\vec{\mathbf{r}})$. Now Taylor expand $F(\varepsilon) = f(\vec{\mathbf{r}}_0 + \varepsilon \cos t \hat{\mathbf{i}}' + \varepsilon \sin t \hat{\mathbf{j}}')$ about $\varepsilon = 0$.

$$\begin{aligned} F(\varepsilon) &= F(0) + F'(0)\varepsilon + O(\varepsilon^2) \\ &= f(\vec{\mathbf{r}}_0) + \varepsilon \cos t \hat{\mathbf{i}}' \cdot \vec{\nabla} f(\vec{\mathbf{r}}_0) + \varepsilon \sin t \hat{\mathbf{j}}' \cdot \vec{\nabla} f(\vec{\mathbf{r}}_0) + O(\varepsilon^2) \\ &= \hat{\mathbf{i}}' \cdot \vec{v}(\vec{\mathbf{r}}_0) + \varepsilon \cos t \hat{\mathbf{i}}' \cdot \vec{\nabla}(\hat{\mathbf{i}}' \cdot \vec{v})(\vec{\mathbf{r}}_0) + \varepsilon \sin t \hat{\mathbf{j}}' \cdot \vec{\nabla}(\hat{\mathbf{i}}' \cdot \vec{v})(\vec{\mathbf{r}}_0) + O(\varepsilon^2) \end{aligned}$$

Similarly, if $G(\varepsilon) = \hat{\mathbf{j}}' \cdot \vec{v}(\vec{\mathbf{r}}_0 + \varepsilon \cos t \hat{\mathbf{i}}' + \varepsilon \sin t \hat{\mathbf{j}}')$,

$$G(\varepsilon) = \hat{\mathbf{j}}' \cdot \vec{v}(\vec{\mathbf{r}}_0) + \varepsilon \cos t \hat{\mathbf{i}}' \cdot \vec{\nabla}(\hat{\mathbf{j}}' \cdot \vec{v})(\vec{\mathbf{r}}_0) + \varepsilon \sin t \hat{\mathbf{j}}' \cdot \vec{\nabla}(\hat{\mathbf{j}}' \cdot \vec{v})(\vec{\mathbf{r}}_0) + O(\varepsilon^2)$$

Hence, the integrand

$$\begin{aligned}
& -\sin t \hat{\mathbf{i}} \cdot \vec{v}(\vec{\mathbf{r}}_0 + \varepsilon \cos t \hat{\mathbf{i}}' + \varepsilon \sin t \hat{\mathbf{j}}') + \cos t \hat{\mathbf{j}} \cdot \vec{v}(\vec{\mathbf{r}}_0 + \varepsilon \cos t \hat{\mathbf{i}}' + \varepsilon \sin t \hat{\mathbf{j}}') \\
&= -\sin t \hat{\mathbf{i}} \cdot \vec{v}(\vec{\mathbf{r}}_0) - \varepsilon \sin t \cos t \hat{\mathbf{i}} \cdot \vec{\nabla}(\hat{\mathbf{i}} \cdot \vec{v})(\vec{\mathbf{r}}_0) - \varepsilon \sin^2 t \hat{\mathbf{j}} \cdot \vec{\nabla}(\hat{\mathbf{i}} \cdot \vec{v})(\vec{\mathbf{r}}_0) \\
&\quad + \cos t \hat{\mathbf{j}} \cdot \vec{v}(\vec{\mathbf{r}}_0) + \varepsilon \cos^2 t \hat{\mathbf{i}} \cdot \vec{\nabla}(\hat{\mathbf{j}} \cdot \vec{v})(\vec{\mathbf{r}}_0) + \varepsilon \sin t \cos t \hat{\mathbf{j}} \cdot \vec{\nabla}(\hat{\mathbf{j}} \cdot \vec{v})(\vec{\mathbf{r}}_0) + O(\varepsilon^2)
\end{aligned}$$

Since

$$\int_0^{2\pi} \sin t \, dt = \int_0^{2\pi} \cos t \, dt = \int_0^{2\pi} \sin t \cos t \, dt = 0 \quad \int_0^{2\pi} \sin^2 t \, dt = \int_0^{2\pi} \cos^2 t \, dt = \pi$$

we have

$$\oint_{C_\varepsilon} \vec{v}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}} = \pi \varepsilon^2 \left[-\hat{\mathbf{j}} \cdot \vec{\nabla}(\hat{\mathbf{i}} \cdot \vec{v})(\vec{\mathbf{r}}_0) + \hat{\mathbf{i}} \cdot \vec{\nabla}(\hat{\mathbf{j}} \cdot \vec{v})(\vec{\mathbf{r}}_0) \right] + O(\varepsilon^3)$$

Sub in

$$\hat{\mathbf{i}}' = -\hat{\mathbf{k}}' \times \hat{\mathbf{j}}' \quad \hat{\mathbf{j}}' = \hat{\mathbf{k}}' \times \hat{\mathbf{i}}' \quad \vec{\nabla} = \sum_{n=1}^3 \hat{\mathbf{i}}_n \frac{\partial}{\partial x_n}$$

where I have renamed the standard basis for \mathbb{R}^3 from $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ to $\hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3$ and the standard coordinates on \mathbb{R}^3 from x, y, z to x_1, x_2, x_3 . Then

$$\begin{aligned}
-\hat{\mathbf{j}} \cdot \vec{\nabla}(\hat{\mathbf{i}} \cdot \vec{v})(\vec{\mathbf{r}}_0) + \hat{\mathbf{i}} \cdot \vec{\nabla}(\hat{\mathbf{j}} \cdot \vec{v})(\vec{\mathbf{r}}_0) &= \sum_{n=1}^3 -\hat{\mathbf{j}} \cdot \hat{\mathbf{i}}_n \frac{\partial \hat{\mathbf{i}} \cdot \vec{v}}{\partial x_n}(\vec{\mathbf{r}}_0) + \sum_{n=1}^3 \hat{\mathbf{i}} \cdot \hat{\mathbf{i}}_n \frac{\partial \hat{\mathbf{j}} \cdot \vec{v}}{\partial x_n}(\vec{\mathbf{r}}_0) \\
&= \sum_{n=1}^3 -(\hat{\mathbf{k}}' \times \hat{\mathbf{i}}') \cdot \hat{\mathbf{i}}_n \frac{\partial \hat{\mathbf{i}} \cdot \vec{v}}{\partial x_n}(\vec{\mathbf{r}}_0) + \sum_{n=1}^3 -(\hat{\mathbf{k}}' \times \hat{\mathbf{j}}') \cdot \hat{\mathbf{i}}_n \frac{\partial \hat{\mathbf{j}} \cdot \vec{v}}{\partial x_n}(\vec{\mathbf{r}}_0) \\
&= \sum_{n=1}^3 -\hat{\mathbf{k}}' \cdot (\hat{\mathbf{i}}' \times \hat{\mathbf{i}}_n) \frac{\partial \hat{\mathbf{i}} \cdot \vec{v}}{\partial x_n}(\vec{\mathbf{r}}_0) + \sum_{n=1}^3 -\hat{\mathbf{k}}' \cdot (\hat{\mathbf{j}}' \times \hat{\mathbf{i}}_n) \frac{\partial \hat{\mathbf{j}} \cdot \vec{v}}{\partial x_n}(\vec{\mathbf{r}}_0) \\
&= \sum_{n=1}^3 \hat{\mathbf{k}}' \cdot (\hat{\mathbf{i}}_n \times \hat{\mathbf{i}}') \frac{\partial \hat{\mathbf{i}} \cdot \vec{v}}{\partial x_n}(\vec{\mathbf{r}}_0) + \sum_{n=1}^3 \hat{\mathbf{k}}' \cdot (\hat{\mathbf{i}}_n \times \hat{\mathbf{j}}') \frac{\partial \hat{\mathbf{j}} \cdot \vec{v}}{\partial x_n}(\vec{\mathbf{r}}_0)
\end{aligned}$$

Since $\hat{\mathbf{k}}' \perp \hat{\mathbf{i}}_n \times \hat{\mathbf{k}}'$, $\hat{\mathbf{k}}' \cdot (\hat{\mathbf{i}}_n \times \hat{\mathbf{k}}') = 0$ and

$$-\hat{\mathbf{j}} \cdot \vec{\nabla}(\hat{\mathbf{i}} \cdot \vec{v})(\vec{\mathbf{r}}_0) + \hat{\mathbf{i}} \cdot \vec{\nabla}(\hat{\mathbf{j}} \cdot \vec{v})(\vec{\mathbf{r}}_0) = \sum_{n=1}^3 \hat{\mathbf{k}}' \cdot \hat{\mathbf{i}}_n \times \frac{\partial}{\partial x_n} [\hat{\mathbf{i}}'(\hat{\mathbf{i}} \cdot \vec{v}) + \hat{\mathbf{j}}'(\hat{\mathbf{j}} \cdot \vec{v}) + \hat{\mathbf{k}}'(\hat{\mathbf{k}} \cdot \vec{v})](\vec{\mathbf{r}}_0)$$

For any orthonormal basis, $\hat{\mathbf{i}}', \hat{\mathbf{j}}', \hat{\mathbf{k}}'$ and any vector \vec{v} ,

$$\hat{\mathbf{i}}'(\hat{\mathbf{i}} \cdot \vec{v}) + \hat{\mathbf{j}}'(\hat{\mathbf{j}} \cdot \vec{v}) + \hat{\mathbf{k}}'(\hat{\mathbf{k}} \cdot \vec{v}) = \vec{v}$$

so

$$-\hat{\mathbf{j}} \cdot \vec{\nabla}(\hat{\mathbf{i}} \cdot \vec{v})(\vec{\mathbf{r}}_0) + \hat{\mathbf{i}} \cdot \vec{\nabla}(\hat{\mathbf{j}} \cdot \vec{v})(\vec{\mathbf{r}}_0) = \sum_{n=1}^3 \hat{\mathbf{k}}' \cdot \hat{\mathbf{i}}_n \times \frac{\partial \vec{v}}{\partial x_n}(\vec{\mathbf{r}}_0) = \hat{\mathbf{k}}' \cdot \vec{\nabla} \times \vec{v}(\vec{\mathbf{r}}_0)$$

and

$$\oint_{C_\varepsilon} \vec{v}(\vec{\mathbf{r}}) \cdot d\vec{\mathbf{r}} = \pi \varepsilon^2 \vec{\nabla} \times \vec{v}(\vec{\mathbf{r}}_0) \cdot \hat{\mathbf{n}} + O(\varepsilon^3)$$