Parametrizing Circles

These notes discuss a simple strategy for parametrizing circles in three dimensions. We start with the circle in the $xy$–plane that has radius $\rho$ and is centred on the origin. This is easy to parametrize:

$$\vec{r}(t) = \rho \cos t \hat{i} + \rho \sin t \hat{j} \quad 0 \leq t \leq 2\pi$$

Note that we can check that $\vec{r}(t)$ lies on the desired circle by checking, firstly, that $\vec{r}(t)$ lies in the correct plane (in this case, the $xy$–plane) and, secondly, that the distance from $\vec{r}(t)$ to the centre of the circle is $\rho$:

$$|\vec{r}(t) - \vec{0}| = |\rho \cos t \hat{i} + \rho \sin t \hat{j}| = \sqrt{(\rho \cos t)^2 + (\rho \sin t)^2} = \rho$$

since $\sin^2 t + \cos^2 t = 1$.

Now let’s move the circle so that its centre is at some general point $\vec{c}$. To parametrize this new circle, which still has radius $\rho$ and which is still parallel to the $xy$–plane, we just translate by $\vec{c}$:

$$\vec{r}(t) = \vec{c} + \rho \cos t \hat{i} + \rho \sin t \hat{j} \quad 0 \leq t \leq 2\pi$$

Finally, let’s consider a circle in general position. The secret to parametrizing a general circle is to replace $\hat{i}$ and $\hat{j}$ by two new vectors $\hat{i}'$ and $\hat{j}'$ which (a) are unit vectors, (b) are parallel to the plane of the desired circle and (c) are mutually perpendicular.

$$\vec{r}(t) = \vec{c} + \rho \cos t \hat{i}' + \rho \sin t \hat{j}' \quad 0 \leq t \leq 2\pi$$

To check that this is correct, observe that

- $\vec{r}(t) - \vec{c}$ is parallel to the plane of the desired circle because $\vec{r}(t) - \vec{c} = \rho \cos t \hat{i}' + \rho \sin t \hat{j}'$ and both $\hat{i}'$ and $\hat{j}'$ are parallel to the plane of the desired circle
- $\vec{r}(t) - \vec{c}$ is of length $\rho$ for all $t$ because
  $$|\vec{r}(t) - \vec{c}|^2 = (\vec{r}(t) - \vec{c}) \cdot (\vec{r}(t) - \vec{c})$$
  $$= (\rho \cos t \hat{i}' + \rho \sin t \hat{j}') \cdot (\rho \cos t \hat{i}' + \rho \sin t \hat{j}')$$
  $$= \rho^2 \cos^2 t \hat{i}' \cdot \hat{i}' + \rho^2 \sin^2 t \hat{j}' \cdot \hat{j}' + 2\rho \cos t \sin t \hat{i}' \cdot \hat{j}'$$
  $$= \rho^2 (\cos^2 t + \sin^2 t) = \rho^2$$

since $\hat{i}' \cdot \hat{i}' = \hat{j}' \cdot \hat{j}' = 1$ ($\hat{i}'$ and $\hat{j}'$ are both unit vectors) and $\hat{i}' \cdot \hat{j}' = 0$ ($\hat{i}'$ and $\hat{j}'$ are perpendicular).
To find such a parametrization in practice, we need to find the centre \( \hat{c} \) of the circle, the radius \( \rho \) of the circle and two mutually perpendicular unit vectors, \( \hat{i}' \) and \( \hat{j}' \), in the plane of the circle. It is often easy to find at least one point \( \vec{p} \) on the circle. Then we can take \( \hat{i}' = \frac{\vec{p} - \vec{c}}{|\vec{p} - \vec{c}|} \). It is also often easy to find a unit vector, \( \hat{k}' \), that is normal to the plane of the circle. Then we can choose \( \hat{j}' = \hat{k}' \times \hat{i}' \).

**Example 1** Let \( C \) be the intersection of the sphere \( x^2 + y^2 + z^2 = 4 \) and the plane \( z = y \). The intersection of any plane with any sphere is a circle. The plane in question passes through the centre of the sphere, so \( C \) has the same centre and same radius as the sphere. So \( C \) has radius 2 and centre \((0, 0, 0)\). The point \((2, 0, 0)\) satisfies both \( x^2 + y^2 + z^2 = 4 \) and \( z = y \) and so is on \( C \). We may choose \( \hat{i}' \) to be the unit vector in the direction from the centre \((0, 0, 0)\) of the circle towards \((2, 0, 0)\). Namely \( \hat{i}' = (1, 0, 0) \). Since the plane of the circle is \( z - y = 0 \), the vector \( \nabla(z - y) = (0, -1, 1) \) is perpendicular to the plane of \( C \). So we may take \( \hat{k}' = \frac{1}{\sqrt{2}}(0, -1, 1) \). Then \( \hat{j}' = \hat{k}' \times \hat{i}' = \frac{1}{\sqrt{2}}(0, -1, 1) \times (1, 0, 0) = \frac{1}{\sqrt{2}}(0, 1, 1) \). Subbing in \( \vec{c} = (0, 0, 0) \), \( \rho = 2 \), \( \hat{i}' = (1, 0, 0) \) and \( \hat{j}' = \frac{1}{\sqrt{2}}(0, 1, 1) \) gives

\[
\vec{r}(t) = 2 \cos t (1, 0, 0) + 2 \sin t \frac{1}{\sqrt{2}}(0, 1, 1) = 2 \left( \cos t, \frac{\sin t}{\sqrt{2}}, \frac{\sin t}{\sqrt{2}} \right) \quad 0 \leq t \leq 2\pi
\]

To check this, note that \( x = 2 \cos t, y = \sqrt{2} \sin t, z = \sqrt{2} \sin t \) satisfies both \( x^2 + y^2 + z^2 = 4 \) and \( z = y \).

**Example 2** Let \( C \) be the circle that passes through the three points \((3, 0, 0)\), \((0, 3, 0)\) and \((0, 0, 3)\). All three points obey \( x + y + z = 3 \). So the circle lies in the plane \( x + y + z = 3 \). We guess, by symmetry, or by looking at the figure below, that the centre of the circle is at the centre of mass of the three points, which is \( \frac{1}{3}[(3, 0, 0) + (0, 3, 0) + (0, 0, 3)] = (1, 1, 1) \). We can check this by checking that \((1, 1, 1)\) is equidistant from the three points:

\[
\begin{align*}
| (3, 0, 0) - (1, 1, 1) | &= | (2, -1, -1) | = \sqrt{6} \\
| (0, 3, 0) - (1, 1, 1) | &= | (-1, 2, -1) | = \sqrt{6} \\
| (0, 0, 3) - (1, 1, 1) | &= | (-1, -1, 2) | = \sqrt{6}
\end{align*}
\]

This tells us both that \((1, 1, 1)\) is indeed the centre and that the radius of \( C \) is \( \sqrt{6} \). We may choose \( \hat{i}' \) to be the unit vector in the direction from the centre \((1, 1, 1)\) of the circle towards \((3, 0, 0)\). Namely \( \hat{i}' = \frac{1}{\sqrt{6}}(2, -1, -1) \). Since the plane of the circle is \( x + y + z = 3 \), the vector \( \nabla(x + y + z) = (1, 1, 1) \) is perpendicular to the plane of \( C \). So we may take \( \hat{k}' = \frac{1}{\sqrt{3}}(1, 1, 1) \). Then \( \hat{j}' = \hat{k}' \times \hat{i}' = \frac{1}{\sqrt{18}}(1, 1, 1) \times (2, -1, -1) = \frac{1}{\sqrt{18}}(0, 3, -3) = \frac{1}{\sqrt{2}}(0, 1, -1) \). Subbing in \( \vec{c} = (1, 1, 1) \), \( \rho = \sqrt{6}, \hat{i}' = \frac{1}{\sqrt{6}}(2, -1, -1) \) and \( \hat{j}' = \frac{1}{\sqrt{2}}(0, 1, -1) \) gives

\[
\vec{r}(t) = (1, 1, 1) + \sqrt{6} \cos t \frac{1}{\sqrt{6}}(2, -1, -1) + \sqrt{3} \sin t \frac{1}{\sqrt{2}}(0, 1, -1)
= (1 + 2 \cos t, 1 - \cos t + \sqrt{3} \sin t, 1 - \cos t - \sqrt{3} \sin t) \quad 0 \leq t \leq 2\pi
\]

To check this, note that \( \vec{r}(0) = (3, 0, 0), \vec{r}(\frac{2\pi}{3}) = (0, 3, 0) \) and \( \vec{r}(\frac{4\pi}{3}) = (0, 0, 3) \) since \( \cos \frac{2\pi}{3} = \cos \frac{4\pi}{3} = -\frac{1}{2}, \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2} \) and \( \sin \frac{4\pi}{3} = -\frac{\sqrt{3}}{2} \).