MATHEMATICS 227 April, 2013 Final Exam Solutions

1. A skier descends the hill $z = \sqrt{4 - x^2 - y^2}$ along a trail with parameterization

$$x = \sin(2\theta), \qquad y = 1 - \cos(2\theta), \qquad z = 2\cos\theta, \qquad 0 \le \theta \le \tfrac{\pi}{2}$$

Let P denote the point on the trail where $x = 1$.
(a) Find the Frenet frame $\hat{\mathbf{T}}, \hat{\mathbf{N}}, \hat{\mathbf{B}}$ and the curvature κ of the ski trail at the point P.
(b) The skier's acceleration at P is $\mathbf{a} = (-2, 3, -2\sqrt{2})$. Find, at P,
  (i) the rate of change of the skier's speed and
  (ii) the skier's velocity (a vector).

Solution. (a) Treating the parameter θ as time–like gives

$$\begin{aligned}
\mathbf{r} &= \big(\sin(2\theta), 1 - \cos(2\theta), 2\cos\theta\big),\\
\mathbf{v} &= \big(2\cos(2\theta), 2\sin(2\theta), -2\sin\theta\big),\\
\mathbf{a} &= \big(-4\sin(2\theta), 4\cos(2\theta), -2\cos\theta\big).
\end{aligned}$$

At the point P, we have $\theta = \pi/4$, giving instantaneous values

$$\mathbf{r} = (1, 1, \sqrt{2}), \quad \mathbf{v} = \big(0, 2, -\sqrt{2}\big), \quad v = |\mathbf{v}| = \sqrt{6}, \quad \mathbf{a} = \big(-4, 0, -\sqrt{2}\big).$$

Hence $\hat{\mathbf{T}} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{6}}\big(0, 2, -\sqrt{2}\big)$.
Now $\hat{\mathbf{B}} = \frac{\mathbf{v}\times\mathbf{a}}{|\mathbf{v}\times\mathbf{a}|} = \frac{1}{\sqrt{26}}\big(-\sqrt{2}, 2\sqrt{2}, 4\big) = \frac{1}{\sqrt{13}}\big(-1, 2, 2\sqrt{2}\big)$, since

$$\mathbf{v}\times\mathbf{a} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 0 & 2 & -\sqrt{2} \\ -4 & 0 & -\sqrt{2} \end{vmatrix} = \big(-2\sqrt{2}, 4\sqrt{2}, 8\big), \qquad |\mathbf{v}\times\mathbf{a}| = \sqrt{104} = 2\sqrt{26}.$$

This leads to

$$\hat{\mathbf{N}} = \hat{\mathbf{B}}\times\hat{\mathbf{T}} = \frac{1}{\sqrt{78}}\begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ -1 & 2 & 2\sqrt{2} \\ 0 & 2 & -\sqrt{2} \end{vmatrix} = -\frac{1}{\sqrt{78}}\big(6\sqrt{2}, \sqrt{2}, 2\big) = -\frac{1}{\sqrt{39}}\big(6, 1, \sqrt{2}\big).$$

Finally,

$$\kappa = \frac{|\mathbf{v}\times\mathbf{a}|}{v^3} = \frac{2\sqrt{26}}{(\sqrt{6})^3} = \frac{2\sqrt{2}\sqrt{13}}{6\sqrt{2}\sqrt{3}} = \frac{\sqrt{13}}{3\sqrt{3}} = \frac{\sqrt{39}}{9}.$$

(b) We use the dot product to extract the tangential and normal components of $\mathbf{a} = \frac{dv}{dt}\hat{\mathbf{T}} + v^2\kappa\hat{\mathbf{N}}$, where $v = |\mathbf{v}|$ is the speed:

$$\frac{dv}{dt} = \mathbf{a}\cdot\hat{\mathbf{T}} = (-2, 3, -2\sqrt{2})\cdot\frac{1}{\sqrt{6}}\big(0, 2, -\sqrt{2}\big) = \frac{1}{\sqrt{6}}[0 + 6 + 4] = \frac{10}{\sqrt{6}} = \frac{5}{3}\sqrt{6}.$$

Similarly, $\mathbf{a}\cdot\hat{\mathbf{N}} = v^2\kappa$, so

$$v^2 = \frac{1}{\kappa}\mathbf{a}\cdot\hat{\mathbf{N}} = -\frac{1}{\sqrt{39}}\frac{9}{\sqrt{39}}(-2, 3, -2\sqrt{2})\cdot\big(6, 1, \sqrt{2}\big) = \frac{13\times 9}{39} = 3.$$

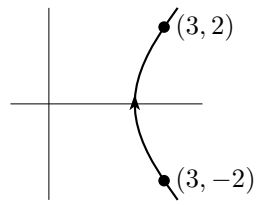
Hence $v = \sqrt{3}$, and $\mathbf{v} = v\hat{\mathbf{T}} = \frac{1}{\sqrt{2}}\big(0, 2, -\sqrt{2}\big) = (0, \sqrt{2}, -1)$.

2. Let $\phi(x,y) = xy$ and let $\mathbf{F} = \boldsymbol{\nabla}\phi$.
(a) Find an equation for the field line of $\mathbf{F}$ which passes through the point $(3,2)$. Sketch it and verify that it also passes through $(3,-2)$.
(b) Find $\int_{(3,-2)}^{(3,2)} \mathbf{F}\cdot d\mathbf{r}$, where the line integral is along the field line of (a).

Solution. (a) The vector field is $\mathbf{F} = \boldsymbol{\nabla}\phi = y\,\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}}$. All field lines have $(dx, dy) \parallel \mathbf{F}$ so that

$$\tfrac{dx}{y} = \tfrac{dy}{x} \iff x\,dx = y\,dy \iff \tfrac{x^2}{2} = \tfrac{y^2}{2} + C$$

To pass through $(3,2)$, the constant C must obey $\frac{3^2}{2} = \frac{2^2}{2} + C$ or $C = \frac{5}{2}$. At the point $(3,2)$, we have $x > 0$. By continuity and the requirement (from the equation) that $x^2 \geq 5$, we have $x > 0$ on the entire flow line. So the equation of the desired field line is $\boxed{x^2 = y^2 + 5,\ x > 0}$. This is a one branch of a hyperbola with asympototes $x = \pm y$. We want the branch of the hyperbola containing $(3,2)$, which is the branch in the sketch below. The point $(3,-2)$ also obeys the equation of the hyperbola and is on the same branch.



(b) Since F is conservative with potential ϕ,

$$\int_{(3,-2)}^{(3,2)} \mathbf{F}\cdot d\mathbf{r} = \phi(3,2) - \phi(3,-2) = 3\times 2 - 3\times(-2) = \boxed{12}$$

3. Find the work $\int_C \mathbf{F}\cdot d\mathbf{r}$ done by the force $\mathbf{F} = 2xy\,\hat{\boldsymbol{\imath}} + zy\,\hat{\boldsymbol{\jmath}} + x^2\,\hat{\mathbf{k}}$ on a particle as it moves from $(2,0,1)$ to $(0,2,5)$ along the curve C of intersection (in the first octant $x,y,z \geq 0$) of the paraboloid $z = (x-1)^2 + y^2$ and the plane $z = 5 - 2x$.

Solution. Parametrize C by

$$\begin{aligned} x(t) &= t \\ z(t) &= 5 - 2x(t) = 5 - 2t \\ y(t) &= \sqrt{z(t) - (x(t)-1)^2} = \sqrt{5 - 2t - (t-1)^2} = \sqrt{4 - t^2} \end{aligned}$$

with t running from 2 to 0. Then

$$\begin{aligned} \mathbf{r}'(t) &= \hat{\boldsymbol{\imath}} - \tfrac{t}{\sqrt{4-t^2}}\hat{\boldsymbol{\jmath}} - 2\hat{\mathbf{k}} \\ \mathbf{F}\big(\mathbf{r}(t)\big)\cdot\mathbf{r}'(t) &= 2t\sqrt{4-t^2} - (5-2t)\sqrt{4-t^2}\,\tfrac{t}{\sqrt{4-t^2}} - 2t^2 = 2t\sqrt{4-t^2} - 5t \\ \int_C \mathbf{F}\cdot d\mathbf{r} &= \int_2^0 \big[2t\sqrt{4-t^2} - 5t\big]\,dt = \Big[-\tfrac{2}{3}(4-t^2)^{3/2} - 5\tfrac{t^2}{2}\Big]_2^0 = -\tfrac{16}{3} + 10 = \boxed{\tfrac{14}{3}} \end{aligned}$$

4. Find the surface area of the torus obtained by rotating the circle $(x-3)^2 + z^2 = 4$, $y = 0$ about the z–axis.

Solution. The torus may be parametrized by

$$x = (3 + 2\cos\phi)\cos\theta \qquad y = (3 + 2\cos\phi)\sin\theta \qquad z = 2\sin\phi \qquad 0 \leq \phi, \theta \leq 2\pi$$

Then

$$\mathbf{T}_\theta = (3+2\cos\phi)\big[-\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}\big] \qquad \mathbf{T}_\phi = 2\big[-\sin\phi\cos\theta\hat{\boldsymbol{\imath}} - \sin\phi\sin\theta\hat{\boldsymbol{\jmath}} + \cos\phi\hat{\mathbf{k}}\big]$$

As $-\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}$ and $-\sin\phi\cos\theta\hat{\boldsymbol{\imath}} - \sin\phi\sin\theta\hat{\boldsymbol{\jmath}} + \cos\phi\hat{\mathbf{k}}$ are mutually perpendicular unit vectors, their cross product is a unit vector and

$$|\mathbf{T}_\theta \times \mathbf{T}_\phi| = 2(3+2\cos\phi) \quad \Longrightarrow \quad dS = 2(3+2\cos\phi)\,d\theta\,d\phi$$

The total surface area of the torus is

$$2\int_0^{2\pi} d\phi \int_0^{2\pi} d\theta\,(3+2\cos\phi) = (2\pi)2\int_0^{2\pi} d\phi\,(3+2\cos\phi) = \boxed{24\pi^2}$$

5. Let S be the portion of the sphere $x^2+y^2+z^2=36$ with $y+z\le 6$. Orient S with normal pointing away from $(0,0,0)$. Let

$$\mathbf{F} = (y^2z - x)\hat{\boldsymbol{\imath}} + (x+y+z)\hat{\boldsymbol{\jmath}} - x\hat{\mathbf{k}}$$

Evaluate $\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS$.

Solution. Let S' be the portion of $y+z=6$ that is inside the sphere $x^2+y^2+z^2=36$ (with $\hat{\mathbf{n}}$ the upward pointing normal) and let V be the portion of the ball $x^2+y^2+z^2\le 36$ with $y+z\le 6$. Then, by the divergence theorem

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iiint_V \boldsymbol{\nabla}\cdot\mathbf{F}\,dV - \iint_{S'} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS$$

As $\boldsymbol{\nabla}\cdot\mathbf{F} = -1+1-0=0$ and, on S', $\hat{\mathbf{n}} = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{\jmath}}+\hat{\mathbf{k}})$ and $y+z=6$

$$\begin{aligned}\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS &= -\iint_{S'} \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = -\tfrac{1}{\sqrt{2}}\iint_{S'}(x+y+z-x)\,dS = -\tfrac{1}{\sqrt{2}}\iint_{S'}(y+z)\,dS = -6\tfrac{1}{\sqrt{2}}\iint_{S'} dS \\ &= -6\tfrac{1}{\sqrt{2}}\ \text{area}(S')\end{aligned}$$

S' is a circular disk. Its center (x_c,y_c,z_c) has to obey $y_c+z_c=6$. By symmetry, $x_c=0$ and $y_c=z_c$, so $y_c=z_c=3$. Any point, like $(0,0,6)$, which satisfies both $y+z=6$ and $x^2+y^2+z^2=36$ is on the boundary of S'. So the radius of S' is $\big|(0,3,3)-(0,0,6)\big| = \big|(0,3,-3)\big| = 3\sqrt{2}$. So the area of S' is 18π and

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = -6\tfrac{1}{\sqrt{2}}\times 18\pi = \boxed{-54\sqrt{2}\pi}$$

6. Let $\mathbf{F} = (e^{x^2}+y)\,\hat{\boldsymbol{\imath}} + (\sin y^3 + xz)\,\hat{\boldsymbol{\jmath}} + z^2\,\hat{\mathbf{k}}$. Evaluate $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$ where $\mathcal{C}$ is the curve $\begin{Bmatrix} x^2+y^2=10 \\ x+y+z=4 \end{Bmatrix}$ with positive (i.e. counter-clockwise) orientation as viewed from high on the z-axis.

Solution. Define

$$S = \big\{\ (x,y,z)\in\mathbb{R}^3 \ \big|\ x+y+z=4,\ x^2+y^2\le 10\ \big\}$$

oriented with $\hat{\mathbf{n}} = \frac{1}{\sqrt{3}}(1,1,1)$. Parametrizing S by $z=4-x-y$, $x^2+y^2\le 10$, we have

$$\begin{aligned}\hat{\mathbf{n}}\,dS &= (1,1,1)\ dx\,dy \\ \boldsymbol{\nabla}\times\mathbf{F} &= -x\hat{\boldsymbol{\imath}} + (z-1)\hat{\mathbf{k}} \\ \boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{n}}\ dS &= (z-x-1)\ dx\,dy = (3-2x-y)\ dx\,dy \quad \text{on } S\end{aligned}$$

So, by Stokes' theorem,

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \iint_S \boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{n}}\,dS = \iint_{x^2+y^2\le 10} (3-2x-y)\;dx\,dy = 3\iint_{x^2+y^2\le 10} dx\,dy = \boxed{30\pi}$$

7. A pile of wet sand having total volume 5π covers the disk $z=0$, $x^2+y^2\le 1$. Call the top surface of the pile of sand S. The momentum of water vapour is given by the vector field

$$\mathbf{F} = \boldsymbol{\nabla}\phi + \mu\boldsymbol{\nabla}\times\mathbf{G}$$

where ϕ is the water concentration, $\mathbf{G}$ is the temperature gradient and μ is a constant. Suppose that $\phi = x^2-y^2+z^2$ and $\mathbf{G} = \frac{1}{3}\big(-y^3\hat{\boldsymbol{\imath}} + x^3\hat{\boldsymbol{\jmath}} + z^3\hat{\mathbf{k}}\big)$. Find the flux of $\mathbf{F}$ upward through S.

Solution 1. Let V be the sand pile. Then the surface of V (with outward normal) is the union of S (with normal pointing away from the origin) and the disk $B = \{\ (x,y,0)\ \mid\ x^2+y^2\le 1\ \}$, with normal $-\hat{\mathbf{k}}$. Since

$$\boldsymbol{\nabla}\phi = 2x\hat{\boldsymbol{\imath}} - 2y\hat{\boldsymbol{\jmath}} + 2z\hat{\mathbf{k}} \qquad \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\phi) = 2 \qquad \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{G}) = 0$$

the Divergence Theorem implies

$$\begin{aligned}\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS &= \iiint_V \boldsymbol{\nabla}\cdot\mathbf{F}\,dV - \iint_B \mathbf{F}\cdot(-\hat{\mathbf{k}})\,dS\\ &= \iiint_V 2\,dV + \iint_B (2z + \mu\hat{\mathbf{k}}\cdot\boldsymbol{\nabla}\times\mathbf{G})\,dS\\ &= 10\pi + \mu\iint_B \hat{\mathbf{k}}\cdot\boldsymbol{\nabla}\times\mathbf{G}\,dS\end{aligned}$$

since V has volume 5π and $z=0$ on B. Since $\hat{\mathbf{k}}\cdot\boldsymbol{\nabla}\times\mathbf{G} = x^2+y^2$ we have

$$\iint_B \hat{\mathbf{k}}\cdot\boldsymbol{\nabla}\times\mathbf{G}\,dS = \iint_B (x^2+y^2)\,dx\,dy = \int_0^1 dr\int_0^{2\pi} d\theta\ r^3 = 2\pi\frac{r^4}{4}\Big|_0^1 = \frac{\pi}{2}$$

All together $\boxed{\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = 10\pi + \mu\frac{\pi}{2}}$.

Solution 2. As in solution 1, we have

$$\iint_S \mathbf{F}\cdot\hat{\mathbf{n}}\,dS = 10\pi + \mu\iint_B \hat{\mathbf{k}}\cdot\boldsymbol{\nabla}\times\mathbf{G}\,dS$$

This time we use Stokes' Theorem or Green's Theorem to give $\iint_B \hat{\mathbf{k}}\cdot\boldsymbol{\nabla}\times\mathbf{G}\,dS = \oint_C \mathbf{G}\cdot d\mathbf{r}$, where C is $x^2+y^2=1$ oriented counterclockwise. Parametrizing C by $x(t)=\cos t,\ y(t)=\sin t,\ z(t)=0$,

$$\begin{aligned}\oint_C \mathbf{G}\cdot d\mathbf{r} &= \frac{1}{3}\int_0^{2\pi}\Big[-\sin^3 t(-\sin t) + \cos^3 t(\cos t)\Big]\,dt = \frac{1}{3}\int_0^{2\pi}\Big[\sin^4 t + \cos^4 t\Big]\,dt = \frac{2}{3}\int_0^{2\pi}\cos^4 t\,dt\\ &= \frac{2}{3}\int_0^{2\pi}\Big[\tfrac{1+\cos 2t}{2}\Big]^2\,dt = \frac{1}{6}\int_0^{2\pi}\big[1+2\cos 2t+\cos^2 2t\big]\,dt = \frac{1}{6}\big[2\pi+0+\pi\big]\,dt = \frac{\pi}{2}\end{aligned}$$

which gives the same answer as solution 1 did.

8. Say whether each of the following statements is true or false and *explain why*. You may assume that the curves, surfaces and functions are all sufficiently smooth.

(a) Let ϕ be a scalar field. Suppose that

$$\iint_S \nabla\phi \cdot \hat{\mathbf{n}}\, dS = 0$$

for every closed oriented surface S. Here $\hat{\mathbf{n}}$ is the outward unit normal to S. Then ϕ is a constant.

(b) If $\mathbf{F}$ and $\mathbf{G}$ are two vector fields defined in $\mathbf{R}^3$ and $\int_S \mathbf{F}\cdot\hat{\mathbf{n}}\, dS = \int_S \mathbf{G}\cdot\hat{\mathbf{n}}\, dS$ for every orientable surface S, then $\mathbf{F} = \mathbf{G}$.

(c) Let $\mathbf{F}$ be a vector field in the xy-plane which is perpendicular to $\mathbf{r} = (x, y)$ at all points. Then each flow line of $\mathbf{F}$ is contained in a circle.

(d) Let $\mathbf{F}$ be a vector field in the xy-plane which is perpendicular to $\mathbf{r} = (x, y)$ at all points. Then the unit circle is a flow line of $\mathbf{F}$.

Solution. (a) $\boxed{\text{False}}$. By the divergence theorem, if V is the solid bounded by S,

$$\iint_S \nabla\phi \cdot \hat{\mathbf{n}}\ dS = \iiint_V \nabla\cdot\nabla\phi\ dV$$

There are many nonconstant scalar fields ϕ for which $\nabla\cdot(\nabla\phi) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$. For example $\phi = x$ or $\phi = x^2 - y^2$.

(b) $\boxed{\text{True}}$. Pick any point $\mathbf{a}$ and any unit vector $\hat{\mathbf{d}}$. I will show that

$$\hat{\mathbf{d}}\cdot\mathbf{F}(\mathbf{a}) = \hat{\mathbf{d}}\cdot\mathbf{G}(\mathbf{a})$$

For each $\epsilon > 0$, let S_ϵ be a flat disk of radius ϵ centered on $\mathbf{a}$ that is normal to $\hat{\mathbf{d}}$. Then

$$0 = \int_{S_\epsilon} \mathbf{F}\cdot\hat{\mathbf{n}}\, dS - \int_{S_\epsilon} \mathbf{G}\cdot\hat{\mathbf{n}}\, dS = \int_{S_\epsilon} (\mathbf{F}-\mathbf{G})\cdot\hat{\mathbf{d}}\ dS$$

If $\hat{\mathbf{d}}\cdot\mathbf{F}(\mathbf{a}) \neq \hat{\mathbf{d}}\cdot\mathbf{G}(\mathbf{a})$, then $\hat{\mathbf{d}}\cdot\big(\mathbf{F}(\mathbf{a})-\mathbf{G}(\mathbf{a})\big) \neq 0$. By continuity, if ϵ is small enough, $\hat{\mathbf{d}}\cdot\big(\mathbf{F}(\mathbf{x})-\mathbf{G}(\mathbf{x})\big)$ is of the same sign and bounded away from zero for all $\mathbf{x}\in S_\epsilon$. Hence $\int_{S_\epsilon}(\mathbf{F}-\mathbf{G})\cdot\hat{\mathbf{d}}\ dS$ is nonzero, which is a contradiction.

(c) $\boxed{\text{True}}$. Suppose that $\mathbf{r}(t)$ is a field line. Then

$$\mathbf{r}'(t) = \mathbf{F}\big(\mathbf{r}(t)\big) \perp \mathbf{r}(t) \quad\Longrightarrow\quad \tfrac{d}{dt}|\mathbf{r}(t)|^2 = 2\mathbf{r}'(t)\cdot\mathbf{r}(t) = 0$$

Hence, $|\mathbf{r}(t)|^2$ is independent of time so that $\mathbf{r}(t)$ lies on a circle centered on the origin.

(d) $\boxed{\text{False}}$. If the vector field is the zero field, then every field line is just a single point.