1. A skier descends the hill \( z = \sqrt{4 - x^2 - y^2} \) along a trail with parameterization

\[
x = \sin(2\theta), \quad y = 1 - \cos(2\theta), \quad z = 2\cos \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}
\]

Let \( P \) denote the point on the trail where \( x = 1 \).
(a) Find the Frenet frame \( \hat{T}, \hat{N}, \hat{B} \) and the curvature \( \kappa \) of the ski trail at the point \( P \).
(b) The skier's acceleration at \( P \) is \( \mathbf{a} = (-2, 3, -2\sqrt{2}) \). Find, at \( P \),

(i) the rate of change of the skier's speed and
(ii) the skier's velocity (a vector).

Solution. (a) Treating the parameter \( \theta \) as time–like gives

\[
\mathbf{r} = (\sin(2\theta), 1 - \cos(2\theta), 2\cos \theta),
\quad \mathbf{v} = (2\cos(2\theta), 2\sin(2\theta), -2\sin \theta),
\quad \mathbf{a} = (-4\sin(2\theta), 4\cos(2\theta), -2\cos \theta).
\]

At the point \( P \), we have \( \theta = \pi/4 \), giving instantaneous values

\[
\mathbf{r} = (1, 1, \sqrt{2}), \quad \mathbf{v} = (0, 2, -\sqrt{2}), \quad v = |\mathbf{v}| = \sqrt{6}, \quad \mathbf{a} = (-4, 0, -\sqrt{2}).
\]

Hence \( \hat{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{\sqrt{6}}(0, 2, -\sqrt{2}) \).

Now \( \hat{B} = \frac{\mathbf{v} \times \mathbf{a}}{|\mathbf{v} \times \mathbf{a}|} = \frac{1}{\sqrt{104}}(-\sqrt{2}, 2\sqrt{2}, 4) = \frac{1}{\sqrt{104}}(-1, 2, 2\sqrt{2}) \), since

\[
\mathbf{v} \times \mathbf{a} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
0 & 2 & -\sqrt{2} \\
-4 & 0 & -\sqrt{2}
\end{vmatrix} = (-2\sqrt{2}, 4\sqrt{2}, 8), \quad |\mathbf{v} \times \mathbf{a}| = \sqrt{104} = 2\sqrt{26}.
\]

This leads to

\[
\hat{N} = \hat{B} \times \hat{T} = \frac{1}{\sqrt{78}} \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
-1 & 2 & 2\sqrt{2} \\
0 & 2 & -\sqrt{2}
\end{vmatrix} = -\frac{1}{\sqrt{78}}(6\sqrt{2}, \sqrt{2}, 2) = -\frac{1}{\sqrt{39}}(6, 1, \sqrt{2}).
\]

Finally,

\[
\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{v^3} = \frac{2\sqrt{26}}{(\sqrt{6})^3} = \frac{2\sqrt{2}}{6\sqrt{2}\sqrt{3}} = \frac{\sqrt{13}}{3\sqrt{3}} = \frac{\sqrt{39}}{9}.
\]

(b) We use the dot product to extract the tangential and normal components of \( \mathbf{a} = \frac{dv}{dt} \hat{T} + v^2 \kappa \hat{N} \), where \( v = |\mathbf{v}| \) is the speed:

\[
\frac{dv}{dt} = \mathbf{a} \cdot \hat{T} = (-2, 3, -2\sqrt{2}) \cdot \frac{1}{\sqrt{6}}(0, 2, -\sqrt{2}) = \frac{1}{\sqrt{6}}[0 + 6 + 4] = \frac{10}{\sqrt{6}} = \frac{5}{3}\sqrt{6}.
\]

Similarly, \( \mathbf{a} \cdot \hat{N} = v^2 \kappa \), so

\[
v^2 = \frac{1}{\kappa} \mathbf{a} \cdot \hat{N} = -\frac{1}{\sqrt{39} \sqrt{39}}(-2, 3, -2\sqrt{2}) \cdot (6, 1, \sqrt{2}) = \frac{13 \times 9}{39} = 3.
\]

Hence \( v = \sqrt{3} \), and \( \mathbf{v} = v \hat{T} = \frac{1}{\sqrt{2}}(0, 2, -\sqrt{2}) = (0, \sqrt{2}, -1) \).
2. Let \( \phi(x, y) = xy \) and let \( \mathbf{F} = \nabla \phi \).
(a) Find an equation for the field line of \( \mathbf{F} \) which passes through the point \((3, 2)\). Sketch it and verify that it also passes through \((3, -2)\).
(b) Find \( \int_{(3,-2)}^{(3,2)} \mathbf{F} \cdot d\mathbf{r} \), where the line integral is along the field line of (a).

**Solution.** (a) The vector field is \( \mathbf{F} = \nabla \phi = y \hat{i} + x \hat{j} \). All field lines have \((dx, dy) \parallel \mathbf{F}\) so that

\[
\frac{dx}{y} = \frac{dy}{x} \iff x \, dx = y \, dy \iff \frac{x^2}{2} = \frac{y^2}{2} + C
\]

To pass through \((3, 2)\), the constant \(C\) must obey \(\frac{3^2}{2} = \frac{2^2}{2} + C\) or \(C = \frac{5}{2}\). At the point \((3, 2)\), we have \(x > 0\). By continuity and the requirement (from the equation) that \(x^2 \geq 5\), we have \(x > 0\) on the entire flow line. So the equation of the desired field line is \(x^2 = y^2 + 5, x > 0\). This is a one branch of a hyperbola with asymptotes \(x = \pm y\). We want the branch of the hyperbola containing \((3, 2)\), which is the branch in the sketch below. The point \((3, -2)\) also obeys the equation of the hyperbola and is on the same branch.

(b) Since \(\mathbf{F}\) is conservative with potential \(\phi\),

\[
\int_{(3,-2)}^{(3,2)} \mathbf{F} \cdot d\mathbf{r} = \phi(3, 2) - \phi(3, -2) = 3 \times 2 - 3 \times (-2) = 12
\]

3. Find the work \(\int_C \mathbf{F} \cdot d\mathbf{r}\) done by the force \(\mathbf{F} = 2xy \hat{i} + zy \hat{j} + x^2 \hat{k}\) on a particle as it moves from \((2, 0, 1)\) to \((0, 2, 5)\) along the curve \(C\) of intersection (in the first octant \(x, y, z \geq 0\)) of the paraboloid \(z = (x - 1)^2 + y^2\) and the plane \(z = 5 - 2x\).

**Solution.** Parametrize \(C\) by

\[
x(t) = t \\
y(t) = \sqrt{z(t) - (x(t) - 1)^2} = \sqrt{5 - 2t - (t - 1)^2} = \sqrt{4 - t^2}
\]

with \(t\) running from 2 to 0. Then

\[
\mathbf{r}'(t) = \mathbf{i} - \frac{t}{\sqrt{4 - t^2}} \mathbf{j} - 2 \mathbf{k}
\]

\[
\mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 2t\sqrt{4 - t^2} - (5 - 2t)\sqrt{4 - t^2} - \frac{t^3}{\sqrt{4 - t^2}} - 2t^2 = 2t\sqrt{4 - t^2} - 5t
\]

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_2^0 \left[ 2t\sqrt{4 - t^2} - 5t \right] dt = \left[ -\frac{2}{3}(4 - t^2)^{3/2} - 5\frac{t^2}{2} \right]_2^0 = -\frac{16}{3} + 10 = \frac{14}{3}
\]

4. Find the surface area of the torus obtained by rotating the circle \((x - 3)^2 + z^2 = 4, y = 0\) about the \(z\)-axis.

**Solution.** The torus may be parametrized by

\[
x = (3 + 2 \cos \phi) \cos \theta \quad y = (3 + 2 \cos \phi) \sin \theta \quad z = 2 \sin \phi \quad 0 \leq \phi, \theta \leq 2\pi
\]
Let 
\[ \nabla \cdot F \] 
be upward pointing normal) and let 
\[ \nabla \times \nabla \times \frac{\partial}{\partial r} \]
be the portion of the ball 
\[ x^2 + y^2 + z^2 = 36 \] 
with normal pointing away from \((0, 0, 0)\). Let 
\[ F = (y^2 z - x) \hat{i} + (x + y + z) \hat{j} - x \hat{k} \]
Evaluate \( \int_S F \cdot \hat{n} dS \).

**Solution.** Let \( S' \) be the portion of the \( y + z = 6 \) that is inside the sphere \( x^2 + y^2 + z^2 = 36 \) (with \( \hat{n} \) the upward pointing normal) and let \( V \) be the portion of the ball \( x^2 + y^2 + z^2 \leq 36 \) with \( y + z \leq 6 \). Then, by the divergence theorem
\[ \int_S F \cdot \hat{n} dS = \int_V \nabla \cdot F dV - \int_{S'} F \cdot \hat{n} dS \]
As \( \nabla \cdot F = -1 + 1 - 0 = 0 \) and, on \( S' \), 
\[ \hat{n} = \frac{1}{\sqrt{2}} \left( \hat{j} + \hat{k} \right) \]
and \( y + z = 6 \)
\[ \int_S F \cdot \hat{n} dS = - \int_{S'} F \cdot \hat{n} dS = - \frac{1}{\sqrt{2}} \int_{S'} (x + y + z - x) dS = - \frac{1}{\sqrt{2}} \int_{S'} (y + z) dS = - \frac{1}{\sqrt{2}} \int_{S'} dS = - \frac{1}{\sqrt{2}} \text{area}(S') \]
\( S' \) is a circular disk. Its center \( (x_c, y_c, z_c) \) has to obey \( y_c + z_c = 6 \). By symmetry, \( x_c = 0 \) and \( y_c = z_c \), so \( y_c = z_c = 3 \). Any point, like \((0, 0, 6)\), which satisfies both \( y + z = 6 \) and \( x^2 + y^2 + z^2 = 36 \) is on the boundary of \( S' \). So the radius of \( S' \) is \( \frac{6}{\sqrt{2}} \text{area}(S') \)
\[ \int_S F \cdot \hat{n} dS = - \frac{1}{\sqrt{2}} \times 18\pi = -\frac{54\sqrt{2}}{2\pi} \]
6. Let \( F = (e^{x^2} + y) \hat{i} + (\sin y^3 + xz) \hat{j} + z^2 \hat{k} \). Evaluate \( \int_C F \cdot d\mathbf{r} \) where \( C \) is the curve \( \begin{cases} x^2 + y^2 = 10 \\ x + y + z = 4 \end{cases} \) with positive (i.e. counter-clockwise) orientation as viewed from high on the z-axis.

**Solution.** Define
\[ S = \left\{ (x, y, z) \in \mathbb{R}^3 \mid x + y + z = 4, \ x^2 + y^2 \leq 10 \right\} \]
on oriented with \( \hat{n} = \frac{1}{\sqrt{3}} (1, 1, 1) \). Parametrizing \( S \) by \( z = 4 - x - y, \ x^2 + y^2 \leq 10 \), we have
\[ \hat{n} dS = (1, 1, 1) \ dx \ dy \]
\[ \nabla \times \mathbf{F} = -x \hat{i} + (z - 1) \hat{k} \]
\[ \nabla \times \mathbf{F} \cdot \hat{n} dS = (z - x - 1) \ dx \ dy = (3 - 2x - y) \ dx \ dy \] on \( S \)
So, by Stokes’ theorem,

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \hat{n} \, dS = \iint_{x^2+y^2 \leq 10} (3 - 2x - y) \, dx \, dy = 3 \iint_{x^2+y^2 \leq 10} \, dx \, dy = 30\pi
\]

7. A pile of wet sand having total volume \(5\pi\) covers the disk \(z = 0, x^2 + y^2 \leq 1\). Call the top surface of the pile of sand \(S\). The momentum of water vapour is given by the vector field

\[\mathbf{F} = \nabla \phi + \mu \nabla \times \mathbf{G}\]

where \(\phi\) is the water concentration, \(\mathbf{G}\) is the temperature gradient and \(\mu\) is a constant. Suppose that \(\phi = x^2 - y^2 + z^2\) and \(\mathbf{G} = \frac{1}{2}(-y\mathbf{i} + x\mathbf{j} + z^3\mathbf{k})\). Find the flux of \(\mathbf{F}\) upward through \(S\).

Solution 1. Let \(V\) be the sand pile. Then the surface of \(V\) (with outward normal) is the union of \(S\) (with normal pointing away from the origin) and the disk \(B = \{(x, y, 0) \mid x^2 + y^2 \leq 1\}\), with normal \(-\hat{k}\). Since

\[\nabla \phi = 2x\hat{i} - 2y\hat{j} + 2z\hat{k} \quad \nabla \cdot (\nabla \phi) = 2 \quad \nabla \cdot (\nabla \times \mathbf{G}) = 0\]

the Divergence Theorem implies

\[
\iint_S \mathbf{F} \cdot \hat{n} \, dS = \iiint_V \nabla \cdot \mathbf{F} \, dV - \iint_B \mathbf{F} \cdot (\hat{k}) \, dS
= \iiint_V 2 \, dV + \iint_B (2z + \mu \hat{k} \cdot \nabla \times \mathbf{G}) \, dS
= 10\pi + \mu \iint_B \hat{k} \cdot \nabla \times \mathbf{G} \, dS
\]

since \(V\) has volume \(5\pi\) and \(z = 0\) on \(B\). Since \(\hat{k} \cdot \nabla \times \mathbf{G} = x^2 + y^2\) we have

\[
\iint_B \hat{k} \cdot \nabla \times \mathbf{G} \, dS = \iint_B (x^2 + y^2) \, dx \, dy = \int_0^1 \, dr \int_0^{2\pi} \, d\theta \ r^3 = 2\pi r^4 \bigg|_0^1 = \frac{\pi}{2}
\]

All together \(\iint_S \mathbf{F} \cdot \hat{n} \, dS = 10\pi + \mu \frac{\pi}{2}\).

Solution 2. As in solution 1, we have

\[
\iint_S \mathbf{F} \cdot \hat{n} \, dS = 10\pi + \mu \iint_B \hat{k} \cdot \nabla \times \mathbf{G} \, dS
\]

This time we use Stokes’ Theorem or Green’s Theorem to give \(\iint_B \hat{k} \cdot \nabla \times \mathbf{G} \, dS = \oint_C \mathbf{G} \cdot d\mathbf{r}\), where \(C\) is \(x^2 + y^2 = 1\) oriented counterclockwise. Parametrizing \(C\) by \(x(t) = \cos t, \ y(t) = \sin t, \ z(t) = 0\),

\[
\oint_C \mathbf{G} \cdot d\mathbf{r} = \frac{1}{3} \int_0^{2\pi} \left[ -\sin^3 t(\sin t) + \cos^3 t(\cos t) \right] \, dt = \frac{1}{3} \int_0^{2\pi} \left[ \sin^4 t + \cos^4 t \right] \, dt = \frac{2}{3} \int_0^{2\pi} \cos^4 t \, dt
= \frac{2}{3} \int_0^{2\pi} \left[ \frac{1 + \cos 2t}{2} \right]^2 \, dt = \frac{1}{6} \int_0^{2\pi} \left[ 1 + 2 \cos 2t + \cos^2 2t \right] \, dt = \frac{1}{6} \left[ 2\pi + 0 + \pi \right] \, dt = \frac{\pi}{2}
\]

which gives the same answer as solution 1 did.
8. Say whether each of the following statements is true or false and explain why. You may assume that the curves, surfaces and functions are all sufficiently smooth.

(a) Let \( \phi \) be a scalar field. Suppose that

\[
\int_S \nabla \phi \cdot \hat{n} \, dS = 0
\]

for every closed oriented surface \( S \). Here \( \hat{n} \) is the outward unit normal to \( S \). Then \( \phi \) is a constant.

(b) If \( F \) and \( G \) are two vector fields defined in \( \mathbb{R}^3 \) and

\[
\int_S F \cdot \hat{n} \, dS = \int_S G \cdot \hat{n} \, dS
\]

for every orientable surface \( S \), then \( F = G \).

(c) Let \( F \) be a vector field in the \( xy \)-plane which is perpendicular to \( r = (x, y) \) at all points. Then each flow line of \( F \) is contained in a circle.

(d) Let \( F \) be a vector field in the \( xy \)-plane which is perpendicular to \( r = (x, y) \) at all points. Then the unit circle is a flow line of \( F \).

Solution. (a) False. By the divergence theorem, if \( V \) is the solid bounded by \( S \),

\[
\int_S \nabla \phi \cdot \hat{n} \, dS = \iiint_V \nabla \cdot \nabla \phi \, dV
\]

There are many nonconstant scalar fields \( \phi \) for which \( \nabla \cdot (\nabla \phi) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \). For example \( \phi = x \) or \( \phi = x^2 - y^2 \).

(b) True. Pick any point \( a \) and any unit vector \( \hat{d} \). I will show that

\[
\hat{d} \cdot F(a) = \hat{d} \cdot G(a)
\]

For each \( \epsilon > 0 \), let \( S_\epsilon \) be a flat disk of radius \( \epsilon \) centered on \( a \) that is normal to \( \hat{d} \). Then

\[
0 = \int_{S_\epsilon} F \cdot \hat{n} \, dS - \int_{S_\epsilon} G \cdot \hat{n} \, dS = \int_{S_\epsilon} (F - G) \cdot \hat{d} \, dS
\]

If \( \hat{d} \cdot F(a) \neq \hat{d} \cdot G(a) \), then \( \hat{d} \cdot (F(a) - G(a)) \neq 0 \). By continuity, if \( \epsilon \) is small enough, \( \hat{d} \cdot (F(x) - G(x)) \) is of the same sign and bounded away from zero for all \( x \in S_\epsilon \). Hence \( \int_{S_\epsilon} (F - G) \cdot \hat{d} \, dS \) is nonzero, which is a contradiction.

(c) True. Suppose that \( r(t) \) is a field line. Then

\[
r'(t) = F(r(t)) \perp r(t) \implies \frac{d}{dt}|r(t)|^2 = 2r'(t) \cdot r(t) = 0
\]

Hence, \( |r(t)|^2 \) is independent of time so that \( r(t) \) lies on a circle centered on the origin.

(d) False. If the vector field is the zero field, then every field line is just a single point.