Solutions to Math 227 Final Exam, April, 2005

1. Let \( z = f(x, y) \) be a \( C^3 \) function on an open set \( D \subset \mathbb{R}^2 \), and assume that all partial derivatives of orders one, two and three equal zero at a point \((a, b) \in D\) except for \( \frac{\partial^3 z}{\partial x \partial y^2}(a, b) = 6 \). Determine whether \((a, b)\) is a local maximum, a local minimum or a saddle point, and give reasons for your answer. Remark: Your reasons do not need to be completely rigorous but they should be at least heuristically convincing.

Solution. Near \((a, b)\),

\[
f(x, y) \approx f(a, b) + \frac{1}{12!} 6(x - a)(y - b)^2
\]

As \((x - a)(y - b)^2\) takes both signs arbitrarily close to \((a, b)\), \( f(x, y) \) has a saddle point there.

2. Use the method of Lagrange multipliers (no credit will be given for any other method) to find the points on the ellipse \( x^2 + 4y^2 = 4 \) which are closest to the point \((1, 0)\).

Hint: Minimize the square of the distance from a point on the ellipse to \((1, 0)\), and be careful to find all four solutions to the equations specified by the method of Lagrange multipliers.

Solution. We are to minimize \((x - 1)^2 + y^2\) subject to the constraint that \( x^2 + 4y^2 = 4 \). So define

\[
L(x, y, \lambda) = (x - 1)^2 + y^2 + \lambda(x^2 + 4y^2 - 4)
\]

Then

\[
0 = L_x = 2(x - 1) + 2\lambda x \implies (1 + \lambda)x = 1
\]

\[
0 = L_y = 2y + 8\lambda y \implies y = 0 \text{ or } \lambda = -\frac{1}{4}
\]

\[
0 = L_\lambda = x^2 + 4y^2 - 4
\]

If \( y = 0 \) then the last equation gives \( x = \pm 2 \).

If \( \lambda = -\frac{1}{4} \), then the first equation gives \( x = \frac{4}{3} \) and the last equation gives \( y^2 = 1 - \frac{1}{4}(\frac{4}{3})^2 = \frac{5}{9} \) or \( y = \pm \frac{\sqrt{5}}{3} \). From the table below, we see that there are two points, namely \((\frac{4}{3}, \pm \frac{\sqrt{5}}{3})\), closest to \((1, 0)\).

<table>
<thead>
<tr>
<th>((x, y))</th>
<th>((x - 1)^2 + y^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>((2, 0))</td>
<td>1</td>
</tr>
<tr>
<td>((-2, 0))</td>
<td>9</td>
</tr>
<tr>
<td>((\frac{4}{3}, \frac{\sqrt{5}}{3}))</td>
<td>(\frac{2}{3})</td>
</tr>
<tr>
<td>((\frac{4}{3}, -\frac{\sqrt{5}}{3}))</td>
<td>(\frac{2}{3})</td>
</tr>
</tbody>
</table>

3. Use the change of variables \( x = u/v, \ y = v \) to evaluate the double integral \( \iint_D y \, dx dy \) where \( D \subset \mathbb{R}^2 \) is the region specified by the inequalities \( y \leq 3, \ y \geq x \) and \( 1 \leq xy \leq 4 \).

Solution. In terms of \( u \) and \( v \), the conditions \( y \leq 3, \ y \geq x \) and \( 1 \leq xy \leq 4 \) become

\[
v \leq 3 \quad v \geq \frac{u}{v} \quad 1 \leq u \leq 4
\]

When \( v > 0 \) these conditions are equivalent to

\[
0 < v \leq 3 \quad u \leq v^2 \quad 1 \leq u \leq 4
\]

and when \( v < 0 \), they are equivalent to

\[
v < 0 \quad u \geq v^2 \quad 1 \leq u \leq 4
\]

The figure on the left below shows the domain \( D \) in the original \((x, y)\) coordinates and the figure on the right shows \( D \) in the new \((u, v)\) coordinates.
5. Find the surface area of the part of the paraboloid

\[ z = 2 - x^2 - y^2 \]
lying above the \( xy \)-plane.

**Solution.** Write the equation of the paraboloid in the form \( F(x, y, z) = z + x^2 + y^2 - 2 = 0 \) or in the form \( z = f(x, y) = 2 - x^2 - y^2 \). Then

\[ \hat{n} dS = \pm (2x, 2y, 1) \, dxdy \quad \implies \quad dS = \sqrt{1 + 4x^2 + 4y^2} \, dxdy \]

The part of the parabola lying above the \( xy \)-plane has \( x^2 + y^2 \leq 2 \). So

\[
\text{surface area} = \iint_{x^2+y^2\leq 2} \sqrt{1+4x^2+4y^2} \, dxdy = \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} vr \, dr \, r \sqrt{1+4r^2} \\
= 2\pi \left[ \frac{1}{12} (1 + 4r^2)^{3/2} \right]_0^{\sqrt{2}} = \frac{\pi}{6} (27 - 1) = \frac{13}{3} \pi
\]

4. Let \( a \) the only allowed values are \( a = 1 \), \( b \) given that \( F \) is conservative.

(a) Find the values of \( a \) and \( b \) such that \( F \) is conservative.

(b) With the values of \( a \) and \( b \) determined in part (a), evaluate the vector line integral \( \int F \cdot dr \) taken over any smooth curve starting at \((0,0)\) and ending at \((1,1)\).

**Solution.** Since

\[
\frac{\partial}{\partial x} [be^{x} \sin (\pi y^2) + x + y] = \frac{\partial}{\partial y} [e^{x} \cos (\pi y^2) + ay - 1] \quad \text{for all } x, y
\]

\[
\iff \quad be^{x} \sin (\pi y^2) + 1 = -2\pi ye^{x} \sin (\pi y^2) + a \quad \text{for all } x, y
\]

\[
\iff \quad a = 1 \text{ and } b = -2\pi
\]

the only allowed values are \( a = 1 \), \( b = -2\pi \).

(b) When \( a = 1 \) and \( b = -2\pi \), \( F = \nabla \varphi \) with \( \varphi = e^{x} \cos (\pi y^2) + xy - x + \frac{1}{2} y^2 \). So

\[
\int F \cdot dr = \left[ e^{x} \cos (\pi y^2) + xy - x + \frac{1}{2} y^2 \right]_{(0,0)}^{(1,1)} = (-e + 1 - 1 + \frac{1}{2}) - (1) = -e - \frac{1}{2}
\]
6. Let \( F = \frac{r}{|r|^2} \) be a vector field in \( \mathbb{R}^3 \), where \( r = (x, y, z) \). If \( D \subset \mathbb{R}^3 \) is an open set containing the origin with a smooth boundary \( \partial D \), then it was shown in class that the Divergence or Gauss’s Theorem does not apply directly for \( F, D \) and \( \partial D \). That is, \( \iint_D F \cdot dS \neq \iiint_D \text{div}(F) \, dV \), where the triple integral must be interpreted as an improper integral since \( F \) is not defined at the origin.

(a) Let \( G = \frac{r}{|r|^2} \) with \( D \) as above. Does the Divergence Theorem apply directly for \( G, D \) and \( \partial D \)? More precisely, is \( \iint_{\partial D} G \cdot dS = \iiint_D \text{div}(G) \, dV \)? Here, as above, the triple integral must be interpreted as an improper integral since \( G \) is not defined at the origin. Give precise reasons for your answer.

(b) Verify your answer to part (a) is correct in the special case where \( D = \{ (x, y, z) \mid x^2 + y^2 + z^2 < a^2 \} \).

**Solution.** As a preliminary calculation, we find

\[
\nabla \cdot G = \nabla \cdot \left[ \frac{(x, y, z)}{x^2 + y^2 + z^2} \right] = \frac{3}{x^2 + y^2 + z^2} - \frac{2(x^2 + y^2 + z^2)}{[x^2 + y^2 + z^2]^2} = \frac{1}{x^2 + y^2 + z^2}
\]

(a) Denote by \( B_\varepsilon = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 < \varepsilon^2 \} \) the (solid) ball of radius \( \varepsilon \) centred on the origin and by \( S_\varepsilon = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = \varepsilon^2 \} \) the sphere of radius \( \varepsilon \) centred on the origin, with outward pointing normal. Note that

\[
\lim_{\varepsilon \to 0} \iint_{S_\varepsilon} G \cdot dS = \lim_{\varepsilon \to 0} \iint_{S_\varepsilon} \frac{x}{|r|^2} \cdot \frac{r}{|r|} \, dS = \frac{1}{a} \iint_{S_\varepsilon} \frac{x}{|r|^2} \, dS = \frac{1}{a} \frac{4\pi}{a} \varepsilon^2 = 0
\]

If \( \varepsilon \) is small enough that \( B_\varepsilon \subset D \), then \( G \) is a smooth vector field on

\( D_\varepsilon = D \setminus B_\varepsilon = \{ (x, y, z) \in D \mid (x, y, z) \notin B_\varepsilon \} \)

and the divergence theorem is directly applicable to \( G, D_\varepsilon \) and \( \partial D_\varepsilon = \partial D \setminus S_\varepsilon \). So, using the definition of \( \iiint_D \text{div}(G) \, dV \) as an improper integral,

\[
\iint_D \text{div}(G) \, dV = \lim_{\varepsilon \to 0} \iint_{D_\varepsilon} \text{div}(G) \, dV = \lim_{\varepsilon \to 0} \left[ \iint_{\partial D} G \cdot dS - \iint_{S_\varepsilon} G \cdot dS \right] = \iint_{\partial D} G \cdot dS
\]

so the divergence theorem does apply directly to \( G, D \) and \( \partial D \).

(b) Using the notation of part (a),

\[
\iint_{\partial D} G \cdot dS = \iint_{S_\varepsilon} \frac{x}{|r|^2} \cdot \frac{r}{|r|} \, dS = \frac{1}{a} \iint_{S_\varepsilon} \frac{x}{|r|^2} \, dS = \frac{1}{a} \frac{4\pi}{a} \varepsilon^2 = 4\pi a
\]

and

\[
\iint_{D_\varepsilon} \text{div}(G) \, dV = \lim_{\varepsilon \to 0} \iint_{D \setminus B_\varepsilon} \text{div}(G) \, dV = \lim_{\varepsilon \to 0} \iint_{B_\varepsilon \setminus B_\varepsilon} \frac{x}{|r|^2} \, dV = \lim_{\varepsilon \to 0} \frac{4\pi}{a} \varepsilon^2 = 4\pi a
\]

7. Verify that Stokes’ Theorem is true in the special case of the vector field \( F = (y, xz, x^2) \) over the triangle with vertices at \( (1, 0, 0), (0, 1, 0) \) and \( (0, 0, 1) \).

**Solution.** Denote by \( L_1 \) the line segment from \( (1, 0, 0) \) to \( (0, 1, 0) \), by \( L_2 \) the line segment from \( (0, 1, 0) \) to \( (0, 0, 1) \), by \( L_3 \) the line segment from \( (0, 0, 1) \) to \( (1, 0, 0) \), and by \( C \) the triangular curve \( L_1 + L_2 + L_3 \). Denote by \( S \) the part of the plane \( x + y + z = 1 \) bounded by the triangle \( C \), with upward pointing unit normal vector \( \hat{n} = \frac{1}{\sqrt{3}}(i + j + k) \).
Computation of $\int_{L_1} \mathbf{F} \cdot d\mathbf{r}$: We parametrize $L_1$ by $\mathbf{r}(t) = (1 - t, t, 0)$, $0 \leq t \leq 1$. Then $\mathbf{r}'(t) = (-1, 1, 0)$ so that
\[
\int_{L_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t, 0, (1 - t)^2) \cdot (-1, 1, 0) dt = -\int_0^1 t dt = -\frac{1}{2}
\]

Computation of $\int_{L_2} \mathbf{F} \cdot d\mathbf{r}$: We parametrize $L_2$ by $\mathbf{r}(t) = (0, 1 - t, t)$, $0 \leq t \leq 1$. Then $\mathbf{r}'(t) = (0, -1, 1)$ so that
\[
\int_{L_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (1 - t, 0, 0) \cdot (0, -1, 1) dt = -\int_0^1 0 dt = 0
\]

Computation of $\int_{L_3} \mathbf{F} \cdot d\mathbf{r}$: We parametrize $L_3$ by $\mathbf{r}(t) = (t, 0, 1 - t)$, $0 \leq t \leq 1$. Then $\mathbf{r}'(t) = (1, 0, -1)$ so that
\[
\int_{L_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (0, t - t^2, t^2) \cdot (1, 0, -1) dt = -\int_0^1 t^2 dt = -\frac{1}{3}
\]

Computation of $\int_C \mathbf{F} \cdot d\mathbf{r}$:
\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{L_1} \mathbf{F} \cdot d\mathbf{r} + \int_{L_2} \mathbf{F} \cdot d\mathbf{r} + \int_{L_3} \mathbf{F} \cdot d\mathbf{r} = -\frac{1}{2} + 0 - \frac{1}{3} = -\frac{5}{6}
\]

Computation of $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, dS$ (method 1): We parametrize $S$ by $\mathbf{r}(s, t) = (1, 0, 0) + s(-1, 1, 0) + t(-1, 0, 1) = (1 - s - t, s, t)$, $s \geq 0$, $t \geq 0$, $s + t \leq 1$. To see that this works, observe that
- $\mathbf{r}(0, 0) = (1, 0, 0)$, $\mathbf{r}(1, 0) = (0, 1, 0)$, $\mathbf{r}(0, 1) = (0, 0, 1)$.
- As $s$ runs from 0 to 1, $\mathbf{r}(s, 0) = (1 - s, s, 0)$ runs from $(1, 0, 0)$ to $(0, 1, 0)$ along $L_1$. (Replace $t$ by $s$ in the parametrization of $L_1$ above.)
- As $t$ runs from 0 to 1, $\mathbf{r}(0, t) = (1 - t, 0, t)$ runs from $(1, 0, 0)$ to $(0, 0, 1)$ along $L_3$. (Replace $t$ by 1 - $t$ in the parametrization of $L_3$ above.)
- If we set $s = 1 - t$ and left $t$ run from 0 to 1, then $\mathbf{r}(1 - t, t) = (0, 1 - t, t)$ runs from $(0, 1, 0)$ to $(0, 0, 1)$ along $L_2$. (See the parametrization of $L_2$ above.)
- $\mathbf{r}(s, t)$ lies on the plane $x + y + z = 1$ for all $s$ and $t$.

Since
\[
\nabla \times \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & z \end{vmatrix} = -2\hat{i} - 2\hat{j} + (z - 1)\hat{k} = (-1 + s + t, -2 + 2s + 2t, t - 1)
\]
and
\[
\frac{\partial \mathbf{r}}{\partial s} = (-1, 1, 0) \quad \frac{\partial \mathbf{r}}{\partial t} = (-1, 0, 1) \quad \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t} = (1, 1, 1) \quad \Longrightarrow \quad \mathbf{n} \, dS = (1, 1, 1) \, ds \, dt
\]
we have
\[
\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, dS = \int_0^1 ds \int_0^{1-s} dt \, (-1 + s + t, -2 + 2s + 2t, t - 1) \cdot (1, 1, 1)
\]
\[
= \int_0^1 ds \int_0^{1-s} dt \, (-4 + 3s + 4t)
\]
\[
= \int_0^1 ds \, [(-4 + 3s)(1 - s) + 2(1 - s)^2]
\]
\[
= \int_0^1 ds \, [-2 + 3s - s^2] = -2 + \frac{3}{2} - \frac{1}{3} = -\frac{5}{6}
\]
as desired.

Computation of $\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} \, dS$ (method 2): We parametrize $S$ by using $x$ and $y$ as parameters. That is, $\mathbf{r}(x, y) = (x, y, 1 - x - y)$. As $(x, y, z)$ runs over the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$,
(0,0,1), the parameters \((x,y)\) runs over the triangle with vertices \((1,0), (0,1), (0,0)\). So the domain is \(\{(x,y) \mid x \geq 0, y \geq 0, x+y \leq 1\}\).

Since

\[
\nabla \times \mathbf{F} = \det \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz & x^2 \end{vmatrix} = -x\hat{i} - 2x\hat{j} + (z-1)\hat{k} \quad \text{and} \quad \hat{n}dS = (1,1,1) \, dx \, dy
\]

we have

\[
\iint_S \nabla \times \mathbf{F} \cdot \hat{n} \, dS = \int_0^1 \int_0^{1-x} dy \, dx \, (-x, -2x, -x-y) \cdot (1,1,1)
\]

\[
= -\int_0^1 \int_0^{1-x} dy \, dx \, (4x+y)
\]

\[
= -\int_0^1 dx \, [4x(1-x) + \frac{1}{2}(1-x)^2]
\]

\[
= -\int_0^1 dx \, [\frac{1}{2} + 3x - \frac{7}{2}x^2] = -[\frac{1}{2} + \frac{3}{2} - \frac{7}{2}] = -\frac{3}{2}
\]

as desired.

8. Let \(\omega = \omega_{(x,y)} = xy \, dx\) be a 1–form on \(\mathbb{R}^2\). According to the discussion in your textbook, the exterior derivative or differential of \(\omega\) is a 2–form on \(\mathbb{R}^2\) defined by the formula \(d\omega = d\omega_{(x,y)} = (ydx + xdy) \wedge dx = x \, dy \wedge dx = -x \, dx \wedge dy\). At a point \((x,y)\), \(d\omega\) operates on pairs of vectors in \(\mathbb{R}^2\). For example, at the point \((2,1)\),

\[
d\omega_{(2,1)} \left( \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = -2 \, dx \wedge dy \left( \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = -2 \det \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} = -2(2) = -4
\]

The purpose of this problem is to ask you to give a more geometric definition of the exterior derivative motivated by the generalized Stokes’ Theorem. In order to keep everything very concrete, give a geometric definition of \(d\omega_{(2,1)} \left( \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)\), involving a limit \(h \to 0\) and an integral of \(\omega\) over a certain closed curve \(C_h\) in \(\mathbb{R}^2\). Then verify by use of your definition that \(d\omega_{(2,1)} \left( \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = -4\). Your discussion should be accompanied by a picture showing all key aspects of your solution.

**Solution.** The point here is that, for any constant 2–form \(a \, dx \wedge dy\) and any two vectors \([x_1, y_1]^t, [x_2, y_2]^t\)

\[
\Omega \left( \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right) = a \, dx \wedge dy \left( \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right) = a \det \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}
\]

is \(a\) times the area of the parallelogram with sides \([x_1, y_1]^t\) and \([x_2, y_2]^t\).

For each \(h > 0\), let

- \(P_h\) be the parallelogram in \(\mathbb{R}^2\) that has one vertex at \((2,1)\) and has sides \(h[2,3]^t\) and \(h[0,1]^t\). (We could also choose to have \(P_h\) centred on \((2,1)\).
- \(L_{1h}\) be the line segment from \((2,1)\) to \((2,1) + h(2,3)\),
- \(L_{2h}\) be the line segment from \((2,1) + h(2,3)\) to \((2,1) + h(2,3) + h(0,1)\),
- \(L_{3h}\) be the line segment from \((2,1) + h(2,3) + h(0,1)\) to \((2,1) + h(0,1)\),
- \(L_{4h}\) be the line segment from \((2,1) + h(0,1)\) to \((2,1)\), and
- \(C_h\) be the curve, \(L_{1h} + L_{2h} + L_{3h} + L_{4h}\), bounding \(P_h\).
By the generalized Stokes' theorem
\[
\int_{C_h} \omega = \int \int_{P_h} d\omega
\]
When \( h \) is very small, \( d\omega \) is approximately the constant 2–form \( -2dx \wedge dy \) on \( P_h \). This constant 2–form gives
\[
\int \int_{P_h} (-2dx \wedge dy) = -2 \text{Area}(P_h) = -2 \det \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} h^2
\]
So I choose to define
\[
d\omega_{(2,1)} \left( \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix} \right) = \lim_{h \to 0} \frac{1}{h^2} \int_{C_h} \omega
\]
**Computation of \( \int_{L_{t_1}} \omega \):** We parametrize \( L_1 \) by \((x(t), y(t)) = (2,1) + th(2,3), 0 \leq t \leq 1. \) Then \( \omega = (2 + 2h)(1 + 3h)(2h dt) \) so that
\[
\int_{L_{t_1}} \omega = \int_0^1 2h \left[ 2 + 8th + 6t^2 h^2 \right] dt = 2h[2 + 4h + 2h^2]
\]
**Computation of \( \int_{L_{t_2}} \omega \):** We parametrize \( L_2 \) by \((x(t), y(t)) = (2,1) + h(2,3) + th(0,1), 0 \leq t \leq 1. \) Then \( \omega = (2 + 2h)(1 + 3h + th)(0 dt) = 0 \) so that
\[
\int_{L_{t_2}} \omega = 0
\]
**Computation of \( \int_{L_{t_3}} \omega \):** We parametrize \( L_3 \) by \((x(t), y(t)) = (2,1) + (1 - t)h(2,3) + h(0,1), 0 \leq t \leq 1. \) Then \( \omega = (2 + 2h - 2th)(1 + 4h - 3th)(-2h dt) \) so that
\[
\int_{L_{t_3}} \omega = \int_0^1 (-2h)[(2 + 2h)(1 + 4h) - 2th(1 + 4h) - 3th(2 + 2h) + 6t^2 h^2] dt
\]
\[
= (-2h)[(2 + 2h)(1 + 4h) - h(1 + 4h) - 3h(1 + h) + 3h^2]
\]
**Computation of \( \int_{L_{t_4}} \omega \):** We parametrize \( L_4 \) by \((x(t), y(t)) = (2,1) + (1 - t)h(0,1), 0 \leq t \leq 1. \) Then \( \omega = (2)(1 + h - th)(0 dt) = 0 \) so that
\[
\int_{L_{t_4}} \omega = 0
\]
**Computation of \( \lim_{h \to 0} \frac{1}{h^2} \int_{C_h} \omega \):**
\[
\lim_{h \to 0} \frac{1}{h^2} \int_{C_h} \omega = \lim_{h \to 0} \frac{1}{h^2} \left[ \int_{L_{t_1}} \omega + \int_{L_{t_2}} \omega + \int_{L_{t_3}} \omega + \int_{L_{t_4}} \omega \right]
\]
\[
= \lim_{h \to 0} \frac{1}{h^2} \left[ 2h[2 + 4h + 2h^2] + (-2h)[(2 + 2h)(1 + 4h) - h(1 + 4h) - 3h(1 + h) + 3h^2] \right]
\]
\[
= \lim_{h \to 0} \frac{2}{h} \left[ 2 + 4h + 2h^2 \right] - \left[ (2 + 2h)(1 + 4h) - h(1 + 4h) - 3h(1 + h) + 3h^2 \right]
\]
\[
= \lim_{h \to 0} \frac{2}{h} \left[ 2 + 4h \right] - \left[ (2 + 10h) - h - 3h \right]
\]
\[
= \lim_{h \to 0} \frac{2}{h} [4h - 6h]
\]
\[
= -4
\]
as desired.