Solutions to Math 227 Final Exam, April, 2005

1. Let $z = f(x, y)$ be a C^3 function on an open set $D \subset \mathbb{R}^2$, and assume that all partial derivatives of orders one, two and three equal zero at a point $(a, b) \in D$ except for $\frac{\partial^3 z}{\partial x \partial y^2}(a, b) = 6$. Determine whether (a, b) is a local maximum, a local minimum or a saddle point, and give reasons for your answer. Remark: Your reasons do not need to be completely rigorous but they should be at least heuristically convincing.

   Solution. Near (a, b),
$$f(x, y) \approx f(a, b) + \tfrac{1}{1!2!}6(x - a)(y - b)^2$$
   As $(x - a)(y - b)^2$ takes both signs arbitrarily close to (a, b), $f(x, y)$ has a saddle point there.

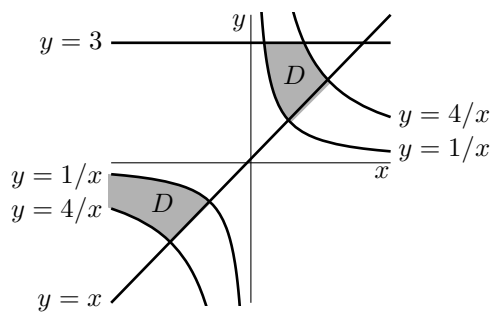
2. Use the method of Lagrange multipliers (no credit will be given for any other method) to find the points on the ellipse $x^2 + 4y^2 = 4$ which are closest to the point $(1, 0)$. *Hint:* Minimize the **square** of the distance from a point on the ellipse to $(1, 0)$, and be careful to find all four solutions to the equations specified by the method of Lagrange multipliers.

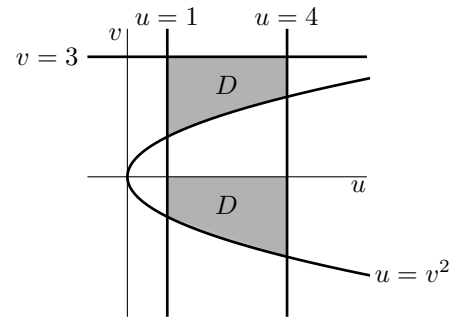
   Solution. We are to minimize $(x - 1)^2 + y^2$ subject to the constraint that $x^2 + 4y^2 - 4 = 0$. So define $L(x, y, \lambda) = (x - 1)^2 + y^2 + \lambda(x^2 + 4y^2 - 4)$. Then
$$\begin{aligned} 0 &= L_x = 2(x - 1) + 2\lambda x &&\Longrightarrow \quad (1 + \lambda)x = 1 \\ 0 &= L_y = 2y + 8\lambda y &&\Longrightarrow \quad y = 0 \text{ or } \lambda = -\tfrac{1}{4} \\ 0 &= L_\lambda = x^2 + 4y^2 - 4 \end{aligned}$$
   If $y = 0$ then the last equation gives $x = \pm 2$.
   If $\lambda = -\frac{1}{4}$, then the first equation gives $x = \frac{4}{3}$ and the last equation gives $y^2 = 1 - \frac{1}{4}\left(\frac{4}{3}\right)^2 = \frac{5}{9}$ or $y = \pm\frac{\sqrt{5}}{3}$. From the table below, we see that there are two points, namely $\left(\frac{4}{3}, \pm\frac{\sqrt{5}}{3}\right)$, closest to $(1, 0)$.

(x, y)	$(x - 1)^2 + y^2$
$(2, 0)$	1
$(-2, 0)$	9
$\left(\frac{4}{3}, \frac{\sqrt{5}}{3}\right)$	$\frac{2}{3}$
$\left(\frac{4}{3}, -\frac{\sqrt{5}}{3}\right)$	$\frac{2}{3}$

3. Use the change of variables $x = u/v$, $y = v$ to evaluate the double integral $\iint_D y \, dxdy$ where $D \subset \mathbb{R}^2$ is the region specified by the inequalities $y \le 3$, $y \ge x$ and $1 \le xy \le 4$.

   Solution. In terms of u and v, the conditions $y \le 3$, $y \ge x$ and $1 \le xy \le 4$ become
$$v \le 3 \qquad v \ge \tfrac{u}{v} \qquad 1 \le u \le 4$$
   When $v > 0$ these conditions are equivalent to
$$0 < v \le 3 \qquad u \le v^2 \qquad 1 \le u \le 4$$
   and when $v < 0$, they are equivalent to
$$v < 0 \qquad u \ge v^2 \qquad 1 \le u \le 4$$
   The figure on the left below shows the domain D in the original (x, y) coordinates and the figure on the right shows D in the new (u, v) coordinates





The inverse change of coordinates is $u = xy$, $v = y$ and

$$\left|\tfrac{\partial(x,y)}{\partial(u,v)}\right| = \left|\det\begin{bmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{bmatrix}\right| = \tfrac{1}{|v|}$$

so that the integral over the upper part of D is

$$\int_1^4 du \int_{\sqrt{u}}^3 dv\ \tfrac{1}{|v|}v = \int_1^4 du \int_{\sqrt{u}}^3 dv = \int_1^4 du\ [3-\sqrt{u}] = \left[3u - \tfrac{2}{3}u^{3/2}\right]_1^4 = 12 - \tfrac{16}{3} - 3 + \tfrac{2}{3} = \tfrac{13}{3}$$

and the integral over the lower part of D (which should be negative since y is negative there) is

$$\int_1^4 du \int_{-\sqrt{u}}^0 dv\ \tfrac{1}{|v|}v = -\int_1^4 du \int_{-\sqrt{u}}^0 dv = -\int_1^4 du\ \sqrt{u} = -\left[\tfrac{2}{3}u^{3/2}\right]_1^4 = -\left[\tfrac{16}{3} - \tfrac{2}{3}\right] = -\tfrac{14}{3}$$

All together, the integral over all of D is $-\frac{1}{13}$.

4. Let $\mathbf{F} = \big(e^x\cos(\pi y^2) + ay - 1\,,\ bye^x\sin(\pi y^2) + x + y\big)$ be a vector field in $\mathbb{R}^2$, where a and b are constants.
   (a) Find the values of a and b such that $\mathbf{F}$ is conservative.
   (b) With the values of a and b determined in part (a), evaluate the vector line integral $\int \mathbf{F}\cdot d\mathbf{r}$ taken over any smooth curve starting at $(0,0)$ and ending at $(1,1)$.

   Solution. (a) Since

$$\begin{aligned} &\tfrac{\partial}{dx}\big[bye^x\sin(\pi y^2) + x + y\big] = \tfrac{\partial}{dy}\big[e^x\cos(\pi y^2) + ay - 1\big] && \text{for all } x, y \\ &\iff bye^x\sin(\pi y^2) + 1 = -2\pi ye^x\sin(\pi y^2) + a && \text{for all } x, y \\ &\iff a = 1 \text{ and } b = -2\pi \end{aligned}$$

   the only allowed values are $a = 1$, $b = -2\pi$.
   (b) When $a = 1$ and $b = -2\pi$, $\mathbf{F} = \boldsymbol{\nabla}\varphi$ with $\varphi = e^x\cos(\pi y^2) + xy - x + \frac{1}{2}y^2$. So

$$\int \mathbf{F}\cdot d\mathbf{r} = \left[e^x\cos(\pi y^2) + xy - x + \tfrac{1}{2}y^2\right]_{(0,0)}^{(1,1)} = (-e + 1 - 1 + \tfrac{1}{2}) - (1) = -e - \tfrac{1}{2}$$

5. Find the surface area of the part of the paraboloid $z = 2 - x^2 - y^2$ lying above the xy–plane.

   Solution. Write the equation of the paraboloid in the form $F(x,y,z) = z + x^2 + y^2 - 2 = 0$ or in the form $z = f(x,y) = 2 - x^2 - y^2$. Then

$$\hat{\mathbf{n}}dS = \pm\big(2x, 2y, 1\big)\ dxdy \quad\Longrightarrow\quad dS = \sqrt{1 + 4x^2 + 4y^2}\ dxdy$$

   The part of the parbola lying above the xy–plane has $x^2 + y^2 \le 2$. So

$$\begin{aligned} \text{surface area} &= \iint_{x^2+y^2\le 2} \sqrt{1 + 4x^2 + 4y^2}\ dxdy = \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} dr\ r\sqrt{1 + 4r^2} \\ &= 2\pi\left[\tfrac{1}{12}(1 + 4r^2)^{3/2}\right]_0^{\sqrt{2}} = \tfrac{\pi}{6}[27 - 1] = \tfrac{13}{3}\pi \end{aligned}$$

6. Let $\mathbf{F} = \frac{\mathbf{r}}{\|\mathbf{r}\|^3}$ be a vector field in $\mathbb{R}^3$, where $\mathbf{r} = (x, y, z)$. If $D \subset \mathbb{R}^3$ is an open set containing the origin with a smooth boundary ∂D, then it was shown in class that the Divergence or Gauss's Theorem does not apply directly for $\mathbf{F}$, D and ∂D. That is $\iint_{\partial D} \mathbf{F} \cdot d\mathbf{S} \neq \iiint_D \operatorname{div}(\mathbf{F})\, dV$, where the triple integral must be interpreted as an improper integral since $\mathbf{F}$ is not defined at the origin.

   (a) Let $\mathbf{G} = \frac{\mathbf{r}}{\|\mathbf{r}\|^2}$ with D as above. Does the Divergence Theorem apply directly for $\mathbf{G}$, D and ∂D? More precisely, is $\iint_{\partial D} \mathbf{G} \cdot d\mathbf{S} = \iiint_D \operatorname{div}(\mathbf{G})\, dV$? Here, as above, the triple integral must be interpreted as an improper integral since $\mathbf{G}$ is not defined at the origin. Give precise reasons for your answer.

   (b) Verify your answer to part (a) is correct in the special case where $D = \left\{ (x, y, z) \;\middle|\; x^2+y^2+z^2 < a^2 \right\}$.

   Solution. As a preliminary calculation, we find

$$\boldsymbol{\nabla} \cdot \mathbf{G} = \nabla \cdot \left[\frac{(x, y, z)}{x^2 + y^2 + z^2} \right] = \frac{3}{x^2 + y^2 + z^2} - \frac{2(x^2 + y^2 + z^2)}{[x^2 + y^2 + z^2]^2} = \frac{1}{x^2 + y^2 + z^2}$$

   (a) Denote by $B_\varepsilon = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 + z^2 < \varepsilon^2 \right\}$ the (solid) ball of radius ε centred on the origin and by $S_\varepsilon = \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 + z^2 = \varepsilon^2 \right\}$ the sphere of radius ε centred on the origin, with outward pointing normal. Note that

$$\lim_{\varepsilon \to 0} \iint_{S_\varepsilon} \mathbf{G} \cdot d\mathbf{S} = \lim_{\varepsilon \to 0} \iint_{S_\varepsilon} \tfrac{\mathbf{r}}{\|\mathbf{r}\|^2} \cdot \tfrac{\mathbf{r}}{\|\mathbf{r}\|}\, dS = \lim_{\varepsilon \to 0} \tfrac{1}{\varepsilon} \iint_{S_\varepsilon} dS = \lim_{\varepsilon \to 0} \tfrac{1}{\varepsilon} 4\pi\varepsilon^2 = 0$$

   If ε is small enough that $B_\varepsilon \subset D$, then $\mathbf{G}$ is a smooth vector field on

$$D_\varepsilon = D \setminus B_\varepsilon = \left\{ (x, y, z) \in D \;\middle|\; (x, y, z) \notin B_\varepsilon \right\}$$

   and the divergence theorem is directly applicable to $\mathbf{G}$, D_ε and $\partial D_\varepsilon = \partial D - S_\varepsilon$. So, using the definition of $\iiint_D \operatorname{div}(\mathbf{G})\, dV$ as an improper integral,

$$\iiint_D \operatorname{div}(\mathbf{G})\, dV = \lim_{\varepsilon \to 0} \iiint_{D_\varepsilon} \operatorname{div}(\mathbf{G})\, dV = \lim_{\varepsilon \to 0} \left[\iint_{\partial D} \mathbf{G} \cdot d\mathbf{S} - \iint_{S_\varepsilon} \mathbf{G} \cdot d\mathbf{S} \right] = \iint_{\partial D} \mathbf{G} \cdot d\mathbf{S}$$

   so the divergence theorem does apply directly to $\mathbf{G}$, D and ∂D.

   (b) Using the notation of part (a),

$$\iint_{\partial D} \mathbf{G} \cdot d\mathbf{S} = \iint_{S_a} \tfrac{\mathbf{r}}{\|\mathbf{r}\|^2} \cdot \tfrac{\mathbf{r}}{\|\mathbf{r}\|}\, dS = \tfrac{1}{a} \iint_{S_a} dS = \tfrac{1}{a} 4\pi a^2 = 4\pi a$$

   and

$$\begin{aligned} \iiint_D \operatorname{div}(\mathbf{G})\, dV &= \lim_{\varepsilon \to 0} \iiint_{D \setminus B_\varepsilon} \operatorname{div}(\mathbf{G})\, dV = \lim_{\varepsilon \to 0} \iiint_{B_a \setminus B_\varepsilon} \tfrac{1}{\|\mathbf{r}\|^2}\, dV = \lim_{\varepsilon \to 0} \int_\varepsilon^a \tfrac{1}{R^2} 4\pi R^2\, dR \\ &= \lim_{\varepsilon \to 0} 4\pi[a - \varepsilon] = 4\pi a \end{aligned}$$

7. Verify that Stokes' Theorem is true in the special case of the vector field $\mathbf{F} = \left(y, xz, x^2\right)$ over the triangle with vertices at $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

   Solution. Denote by L_1 the line segment from $(1, 0, 0)$ to $(0, 1, 0)$, by L_2 the line segment from $(0, 1, 0)$ to $(0, 0, 1)$, by L_3 the line segment from $(0, 0, 1)$ to $(1, 0, 0)$, and by C the triangular curve $L_1 + L_2 + L_3$. Denote by S the part of the plane $x + y + z = 1$ bounded by the triangle C, with upward pointing unit normal vector $\hat{\mathbf{n}} = \frac{1}{\sqrt{3}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}})$.

Computation of $\int_{L_1} \mathbf{F}\cdot d\mathbf{r}$*:* We parametrize L_1 by $\mathbf{r}(t) = (1-t, t, 0)$, $0 \le t \le 1$. Then $\mathbf{r}'(t) = (-1, 1, 0)$ so that

$$\int_{L_1} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \big(t, 0, (1-t)^2\big)\cdot(-1,1,0)dt = -\int_0^1 t\ dt = -\tfrac{1}{2}$$

Computation of $\int_{L_2} \mathbf{F}\cdot d\mathbf{r}$*:* We parametrize L_2 by $\mathbf{r}(t) = (0, 1-t, t)$, $0 \le t \le 1$. Then $\mathbf{r}'(t) = (0, -1, 1)$ so that

$$\int_{L_1} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \big(1-t, 0, 0\big)\cdot(0,-1,1)dt = -\int_0^1 0\ dt = 0$$

Computation of $\int_{L_3} \mathbf{F}\cdot d\mathbf{r}$*:* We parametrize L_3 by $\mathbf{r}(t) = (t, 0, 1-t)$, $0 \le t \le 1$. Then $\mathbf{r}'(t) = (1, 0, -1)$ so that

$$\int_{L_1} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \big(0, t-t^2, t^2\big)\cdot(1,0,-1)dt = -\int_0^1 t^2\ dt = -\tfrac{1}{3}$$

Computation of $\int_C \mathbf{F}\cdot d\mathbf{r}$*:*

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{L_1} \mathbf{F}\cdot d\mathbf{r} + \int_{L_2} \mathbf{F}\cdot d\mathbf{r} + \int_{L_3} \mathbf{F}\cdot d\mathbf{r} = -\tfrac{1}{2} + 0 - \tfrac{1}{3} = -\tfrac{5}{6}$$

Computation of $\iint_S \boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{n}}\,dS$ *(method 1):* We parametrize S by $\mathbf{r}(s,t) = (1,0,0) + s(-1,1,0) + t(-1,0,1) = (1-s-t, s, t)$, $s \ge 0$, $t \ge 0$, $s+t \le 1$. To see that this works, observe that

- $\mathbf{r}(0,0) = (1,0,0)$, $\mathbf{r}(1,0) = (0,1,0)$, $\mathbf{r}(0,1) = (0,0,1)$.
- As s runs from 0 to 1, $\mathbf{r}(s,0) = (1-s, s, 0)$ runs from $(1,0,0)$ to $(0,1,0)$ along L_1. (Replace t by s in the parametrization of L_1 above.)
- As t runs from 0 to 1, $\mathbf{r}(0,t) = (1-t, 0, t)$ runs from $(1,0,0)$ to $(0,0,1)$ along L_3. (Replace t by $1-t$ in the parametrization of L_3 above).
- If we set $s = 1-t$ and left t run from 0 to 1, then $\mathbf{r}(1-t, t) = (0, 1-t, t)$ runs from $(0,1,0)$ to $(0,0,1)$ along L_2. (See the parametrization of L_2 above.)
- $\mathbf{r}(s,t)$ lies on the plane $x+y+z=1$ for all s and t.

Since

$$\boldsymbol{\nabla}\times\mathbf{F} = \det\begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{dx} & \frac{\partial}{dy} & \frac{\partial}{dz} \\ y & xz & x^2 \end{vmatrix} = -x\hat{\boldsymbol{\imath}} - 2x\hat{\boldsymbol{\jmath}} + (z-1)\hat{\mathbf{k}} = (-1+s+t\,,\ -2+2s+2t\,,\ t-1)$$

and

$$\tfrac{\partial\mathbf{r}}{\partial s} = (-1,1,0) \qquad \tfrac{\partial\mathbf{r}}{\partial t} = (-1,0,1) \qquad \tfrac{\partial\mathbf{r}}{\partial s}\times\tfrac{\partial\mathbf{r}}{\partial t} = (1,1,1) \quad\Longrightarrow\quad \hat{\mathbf{n}}dS = (1,1,1)\,ds\,dt$$

we have

$$\begin{aligned}\iint_S \boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{n}}\,dS &= \int_0^1 ds\int_0^{1-s} dt\ (-1+s+t\,,\ -2+2s+2t\,,\ t-1)\cdot(1,1,1) \\ &= \int_0^1 ds\int_0^{1-s} dt\ (-4+3s+4t) \\ &= \int_0^1 ds\ [(-4+3s)(1-s) + 2(1-s)^2)] \\ &= \int_0^1 ds\ [-2+3s-s^2] = -2+\tfrac{3}{2}-\tfrac{1}{3} = -\tfrac{5}{6}\end{aligned}$$

as desired.

Computation of $\iint_S \boldsymbol{\nabla}\times\mathbf{F}\cdot\hat{\mathbf{n}}\,dS$ *(method 2):* We parametrize S by using x and y as parameters. That is, $\mathbf{r}(x,y) = (x, y, 1-x-y)$. As (x,y,z) runs over the triangle with vertices $(1,0,0)$, $(0,1,0)$,

$(0,0,1)$, the parameters (x,y) runs over the triangle with vertices $(1,0)$, $(0,1)$, $(0,0)$. So the domain is $\{\ (x,y)\ \big|\ x \geq 0,\ y \geq 0,\ x+y \leq 1\ \}$.
Since

$$\boldsymbol{\nabla} \times \mathbf{F} = \det \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ \frac{\partial}{dx} & \frac{\partial}{dy} & \frac{\partial}{dz} \\ y & xz & x^2 \end{vmatrix} = -x\hat{\boldsymbol{\imath}} - 2x\hat{\boldsymbol{\jmath}} + (z-1)\hat{\mathbf{k}} \qquad \text{and} \qquad \hat{\mathbf{n}}dS = (1,1,1)\,dx\,dy$$

we have

$$\begin{aligned} \iint_S \boldsymbol{\nabla} \times \mathbf{F} \cdot \hat{\mathbf{n}}\, dS &= \int_0^1 dx \int_0^{1-x} dy\ (-x\,,\ -2x\,,\ -x-y) \cdot (1,1,1) \\ &= -\int_0^1 dx \int_0^{1-x} dy\ (4x+y) \\ &= -\int_0^1 dx\ [4x(1-x) + \tfrac{1}{2}(1-x)^2] \\ &= -\int_0^1 dx\ [\tfrac{1}{2} + 3x - \tfrac{7}{2}x^2] = -[\tfrac{1}{2} + \tfrac{3}{2} - \tfrac{7}{6}] = -\tfrac{5}{6} \end{aligned}$$

as desired.

8. Let $\omega = \omega_{(x,y)} = xy\,dx$ be a 1–form on $\mathbb{R}^2$. According to the discussion in your textbook, the exterior derivative or differential of ω is a 2–form on $\mathbb{R}^2$ deffined by the formula $d\omega = d\omega_{(x,y)} = (ydx + xdy) \wedge dx = x\,dy \wedge dx = -x\,dx \wedge dy)$. At a point (x,y), $d\omega$ operates on pairs of vectors in $\mathbb{R}^2$. For example, at the point $(2,1)$,

$$d\omega_{(2,1)}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = -2\ dx \wedge dy\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = -2 \det \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} = -2(2) = -4$$

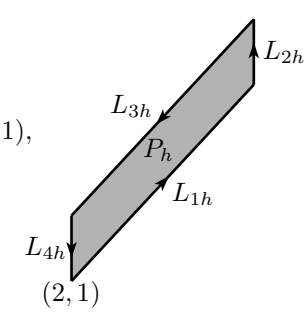
The purpose of this problem is to ask you to give a more geometric definition of the exterior derivative motivated by the generalized Stokes's Theorem. In order to keep everything very concrete, give a geometric definition of $d\omega_{(2,1)}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$, involving a limit $h \to 0$ and an integral of ω over a certain closed curve C_h in $\mathbb{R}^2$. Then verify by use of your definition that $d\omega_{(2,1)}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = -4$. Your discussion should be accompanied by a picture showing all key aspects of your solution.

Solution. The point here is that, for any constant 2–form $a\,dx \wedge dy$ and any two vectors $[x_1, y_1]^t$, $[x_2, y_2]^t$

$$\Omega\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right) = a\ dx \wedge dy\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \right) = a \det \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$$

is a times the area of the parallelogram with sides $[x_1, y_1]^t$ and $[x_2, y_2]^t$.
For each $h > 0$, let

- P_h be the parallelogram in $\mathbb{R}^2$ that has one vertex at $(2,1)$ and has sides $h[2,3]^t$ and $h[0,1]^t$. (We could also choose to have P_h centred on $(2,1)$.)
- L_{1h} be the line segment from $(2,1)$ to $(2,1) + h(2,3)$,
- L_{2h} be the line segment from $(2,1) + h(2,3)$ to $(2,1) + h(2,3) + h(0,1)$,
- L_{3h} be the line segment from $(2,1) + h(2,3) + h(0,1)$ to $(2,1) + h(0,1)$,
- L_{4h} be the line segement from $(2,1) + h(0,1)$ to $(2,1)$, and
- C_h be the curve, $L_{1h} + L_{2h} + L_{3h} + L_{4h}$, bounding P_h.



By the generalized Stokes' theorem

$$\int_{C_h} \omega = \iint_{P_h} d\omega$$

When h is very small, $d\omega$ is approximately the constant 2–form $-2dx \wedge dy$ on P_h. This constant 2–form gives

$$\iint_{P_h} (-2dx \wedge dy) = -2\text{Area}(P_h) = -2 \det \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} h^2$$

So I choose to define

$$d\omega_{(2,1)}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \lim_{h \to 0} \tfrac{1}{h^2} \int_{C_h} \omega$$

Computation of $\int_{L_{1h}} \omega$*:* We parametrize L_1 by $\big(x(t), y(t)\big) = (2,1) + th(2,3)$, $0 \le t \le 1$. Then $\omega = (2+2th)(1+3th)(2h\,dt)$ so that

$$\int_{L_{1h}} \omega = \int_0^1 2h\big[2 + 8th + 6t^2h^2\big]\ dt = 2h[2 + 4h + 2h^2]$$

Computation of $\int_{L_{2h}} \omega$*:* We parametrize L_2 by $\big(x(t), y(t)\big) = (2,1) + h(2,3) + th(0,1)$, $0 \le t \le 1$. Then $\omega = (2+2h)(1+3h+th)(0\,dt) = 0$ so that

$$\int_{L_{2h}} \omega = 0$$

Computation of $\int_{L_{3h}} \omega$*:* We parametrize L_3 by $\big(x(t), y(t)\big) = (2,1) + (1-t)h(2,3) + h(0,1)$, $0 \le t \le 1$. Then $\omega = (2+2h-2th)(1+4h-3th)(-2h\,dt)$ so that

$$\begin{aligned} \int_{L_{3h}} \omega &= \int_0^1 (-2h)\big[(2+2h)(1+4h) - 2th(1+4h) - 3th(2+2h) + 6t^2h^2\big]\ dt \\ &= (-2h)\big[(2+2h)(1+4h) - h(1+4h) - 3h(1+h) + 3h^2\big] \end{aligned}$$

Computation of $\int_{L_{4h}} \omega$*:* We parametrize L_4 by $\big(x(t), y(t)\big) = (2,1) + (1-t)h(0,1)$, $0 \le t \le 1$. Then $\omega = (2)(1+h-th)(0\,dt) = 0$ so that

$$\int_{L_{4h}} \omega = 0$$

Computation of $\lim_{h\to 0} \frac{1}{h^2} \int_{C_h} \omega$*:*

$$\begin{aligned} \lim_{h \to 0} \tfrac{1}{h^2} \int_{C_h} \omega &= \lim_{h \to 0} \tfrac{1}{h^2} \left[\int_{L_{1h}} \omega + \int_{L_{2h}} \omega + \int_{L_{3h}} \omega + \int_{L_{4h}} \omega \right] \\ &= \lim_{h \to 0} \tfrac{1}{h^2} \Big[2h[2 + 4h + 2h^2] + (-2h)\big[(2+2h)(1+4h) - h(1+4h) - 3h(1+h) + 3h^2\big] \Big] \\ &= \lim_{h \to 0} \tfrac{2}{h} \Big[[2 + 4h + 2h^2] - \big[(2+2h)(1+4h) - h(1+4h) - 3h(1+h) + 3h^2\big] \Big] \\ &= \lim_{h \to 0} \tfrac{2}{h} \Big[[2 + 4h] - \big[(2+10h) - h - 3h\big] \Big] \\ &= \lim_{h \to 0} \tfrac{2}{h} \Big[4h - 6h \Big] \\ &= -4 \end{aligned}$$

as desired.