

Multivariable Calculus Warnings

These notes highlight a number of common, but serious, multivariable calculus errors.

Warning 1. A line in three dimensions has infinitely many normal vectors.

Discussion. For example, the line

$$(x, y, z) = (1, 1, 0) + t(1, 2, -2)$$

has direction vector $(1, 2, -2)$. Any vector perpendicular to $(1, 2, -2)$ is perpendicular to the line. The vector (n_1, n_2, n_3) is perpendicular to $(1, 2, -2)$ if and only if

$$0 = (1, 2, -2) \cdot (n_1, n_2, n_3) = n_1 + 2n_2 - 2n_3$$

There is a whole plane of (n_1, n_2, n_3) 's obeying this condition, of which $(2, -1, 0)$, $(0, 1, 1)$ and $(2, 0, 1)$ are only three examples.

Warning 2. The cross product has two properties that are very different from the corresponding properties for the multiplication of real numbers.

$$\begin{aligned}\vec{a} \times \vec{b} &\neq \vec{b} \times \vec{a} \\ \vec{a} \times (\vec{b} \times \vec{c}) &\neq (\vec{a} \times \vec{b}) \times \vec{c}\end{aligned}$$

for most \vec{a} , \vec{b} and \vec{c} . For example

$$\begin{aligned}\hat{i} \times (\hat{i} \times \hat{j}) &= \hat{i} \times \hat{k} = -\hat{k} \times \hat{i} = -\hat{j} \\ (\hat{i} \times \hat{i}) \times \hat{j} &= \vec{0} \times \hat{j} = \vec{0}\end{aligned}$$