

# Equality of Mixed Partial

**Theorem.** *If the partial derivatives  $\frac{\partial^2 f}{\partial x \partial y}$  and  $\frac{\partial^2 f}{\partial y \partial x}$  exist and are continuous at  $(x_0, y_0)$ , then*

$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$$

**Proof:** Here is an outline of the proof. The details are given as footnotes at the end of the outline. Fix  $x_0$  and  $y_0$  and define<sup>(1)</sup>

$$F(h, k) = \frac{1}{hk} [f(x_0 + h, y_0 + k) - f(x_0, y_0 + k) - f(x_0 + h, y_0) + f(x_0, y_0)]$$

Then, by the mean value theorem,

$$\begin{aligned} F(h, k) &\stackrel{2}{=} \frac{1}{h} \left[ \frac{\partial f}{\partial y}(x_0 + h, y_0 + \theta_1 k) - \frac{\partial f}{\partial y}(x_0, y_0 + \theta_1 k) \right] \\ &\stackrel{3}{=} \frac{\partial}{\partial x} \frac{\partial f}{\partial y}(x_0 + \theta_2 h, y_0 + \theta_1 k) \\ F(h, k) &\stackrel{4}{=} \frac{1}{k} \left[ \frac{\partial f}{\partial x}(x_0 + \theta_3 h, y_0 + k) - \frac{\partial f}{\partial x}(x_0 + \theta_3 h, y_0) \right] \\ &\stackrel{5}{=} \frac{\partial}{\partial y} \frac{\partial f}{\partial x}(x_0 + \theta_3 h, y_0 + \theta_4 k) \end{aligned}$$

for some  $0 < \theta_1, \theta_2, \theta_3, \theta_4 < 1$ . All of  $\theta_1, \theta_2, \theta_3, \theta_4$  depend on  $x_0, y_0, h, k$ . Hence

$$\frac{\partial}{\partial x} \frac{\partial f}{\partial y}(x_0 + \theta_2 h, y_0 + \theta_1 k) = \frac{\partial}{\partial y} \frac{\partial f}{\partial x}(x_0 + \theta_3 h, y_0 + \theta_4 k)$$

for all  $h$  and  $k$ . Taking the limit  $(h, k) \rightarrow (0, 0)$  and using the assumed continuity of both partial derivatives at  $(x_0, y_0)$  gives

$$\frac{\partial}{\partial x} \frac{\partial f}{\partial y}(x_0, y_0) = \frac{\partial}{\partial y} \frac{\partial f}{\partial x}(x_0, y_0)$$

## The Details

(1) We define  $F(h, k)$  in this way because both partial derivatives  $\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$  and  $\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$  are defined as limits of  $F(h, k)$  as  $h, k \rightarrow 0$ . For example,

$$\begin{aligned} \lim_{h \rightarrow 0} F(h, k) &= \frac{1}{k} \left[ \frac{\partial f}{\partial x}(x_0, y_0 + k) - \frac{\partial f}{\partial x}(x_0, y_0) \right] \\ \implies \lim_{k \rightarrow 0} \lim_{h \rightarrow 0} F(h, k) &= \lim_{k \rightarrow 0} \frac{1}{k} \left[ \frac{\partial f}{\partial x}(x_0, y_0 + k) - \frac{\partial f}{\partial x}(x_0, y_0) \right] \\ &= \frac{\partial}{\partial y} \frac{\partial f}{\partial x}(x_0, y_0) \end{aligned}$$

Similarly,

$$\begin{aligned} \lim_{k \rightarrow 0} F(h, k) &= \frac{1}{h} \left[ \frac{\partial f}{\partial y}(x_0 + h, y_0) - \frac{\partial f}{\partial y}(x_0, y_0) \right] \\ \implies \lim_{h \rightarrow 0} \lim_{k \rightarrow 0} F(h, k) &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\partial f}{\partial y}(x_0 + h, y_0) - \frac{\partial f}{\partial y}(x_0, y_0) \right] \\ &= \frac{\partial}{\partial x} \frac{\partial f}{\partial y}(x_0, y_0) \end{aligned}$$

(2) Define  $G(y) = f(x_0 + h, y) - f(x_0, y)$ . By the mean value theorem

$$\begin{aligned} F(h, k) &= \frac{1}{h} \left[ \frac{G(y_0 + k) - G(y_0)}{k} \right] \\ &= \frac{1}{h} \frac{dG}{dy}(y_0 + \theta_1 k) \quad \text{for some } 0 < \theta_1 < 1 \\ &= \frac{1}{h} \left[ \frac{\partial f}{\partial y}(x_0 + h, y_0 + \theta_1 k) - \frac{\partial f}{\partial y}(x_0, y_0 + \theta_1 k) \right] \end{aligned}$$

(3) Define  $H(x) = \frac{\partial f}{\partial y}(x, y_0 + \theta_1 k)$ . By the mean value theorem

$$\begin{aligned} F(h, k) &= \frac{1}{h} \left[ H(x_0 + h) - H(x_0) \right] \\ &= \frac{dH}{dx}(x_0 + \theta_2 h) \quad \text{for some } 0 < \theta_2 < 1 \\ &= \frac{\partial}{\partial x} \frac{\partial f}{\partial y}(x_0 + \theta_2 h, y_0 + \theta_1 k) \end{aligned}$$

(4) Define  $A(x) = f(x, y_0 + k) - f(x, y_0)$ . By the mean value theorem

$$\begin{aligned} F(h, k) &= \frac{1}{k} \left[ \frac{A(x_0 + h) - A(x_0)}{h} \right] \\ &= \frac{1}{k} \frac{dA}{dx}(x_0 + \theta_3 h) \quad \text{for some } 0 < \theta_3 < 1 \\ &= \frac{1}{k} \left[ \frac{\partial f}{\partial x}(x_0 + \theta_3 h, y_0 + k) - \frac{\partial f}{\partial x}(x_0 + \theta_3 h, y_0) \right] \end{aligned}$$

(5) Define  $B(y) = \frac{\partial f}{\partial x}(x_0 + \theta_3 h, y)$ . By the mean value theorem

$$\begin{aligned} F(h, k) &= \frac{1}{k} \left[ B(y_0 + k) - B(y_0) \right] \\ &= \frac{dB}{dy}(y_0 + \theta_4 k) \quad \text{for some } 0 < \theta_4 < 1 \\ &= \frac{\partial}{\partial y} \frac{\partial f}{\partial x}(x_0 + \theta_3 h, y_0 + \theta_4 k) \end{aligned}$$

■

**Example.** Here is an example which shows that it is **not always true** that  $\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$ . Define

$$f(x, y) = \begin{cases} xy \frac{x^2 - y^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

This function is continuous everywhere. We now compute the first order partial derivatives.  
 For  $(x, y) \neq (0, 0)$

$$\begin{aligned}\frac{\partial f}{\partial x}(x, y) &= y \frac{x^2 - y^2}{x^2 + y^2} + xy \frac{2x}{x^2 + y^2} - xy \frac{2x(x^2 - y^2)}{(x^2 + y^2)^2} = y \frac{x^2 - y^2}{x^2 + y^2} + xy \frac{4xy^2}{(x^2 + y^2)^2} \\ \frac{\partial f}{\partial y}(x, y) &= x \frac{x^2 - y^2}{x^2 + y^2} - xy \frac{2y}{x^2 + y^2} - xy \frac{2y(x^2 - y^2)}{(x^2 + y^2)^2} = x \frac{x^2 - y^2}{x^2 + y^2} - xy \frac{4yx^2}{(x^2 + y^2)^2}\end{aligned}$$

For  $(x, y) = (0, 0)$

$$\begin{aligned}\frac{\partial f}{\partial x}(0, 0) &= \left[ \frac{d}{dx} f(x, 0) \right]_{x=0} = \left[ \frac{d}{dx} 0 \right]_{x=0} = 0 \\ \frac{\partial f}{\partial y}(0, 0) &= \left[ \frac{d}{dy} f(0, y) \right]_{y=0} = \left[ \frac{d}{dy} 0 \right]_{y=0} = 0\end{aligned}$$

By way of summary, the two first order partial derivatives are

$$\begin{aligned}f_x(x, y) &= \begin{cases} y \frac{x^2 - y^2}{x^2 + y^2} + \frac{4x^2 y^3}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \\ f_y(x, y) &= \begin{cases} x \frac{x^2 - y^2}{x^2 + y^2} - \frac{4x^3 y^2}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}\end{aligned}$$

Both  $\frac{\partial f}{\partial x}(x, y)$  and  $\frac{\partial f}{\partial y}(x, y)$  are continuous. Finally, we compute

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial y}(0, 0) &= \left[ \frac{d}{dx} f_y(x, 0) \right]_{x=0} = \lim_{h \rightarrow 0} \frac{1}{h} [f_y(h, 0) - f_y(0, 0)] = \lim_{h \rightarrow 0} \frac{1}{h} [h \frac{h^2 - 0^2}{h^2 + 0^2} - 0] = 1 \\ \frac{\partial^2 f}{\partial y \partial x}(0, 0) &= \left[ \frac{d}{dy} f_x(0, y) \right]_{y=0} = \lim_{k \rightarrow 0} \frac{1}{k} [f_x(0, k) - f_x(0, 0)] = \lim_{k \rightarrow 0} \frac{1}{k} [k \frac{0^2 - k^2}{0^2 + k^2} - 0] = -1\end{aligned}$$