

# Equality of Mixed Partialials

**Theorem.** *If the partial derivatives  $\frac{\partial^2 f}{\partial x \partial y}$  and  $\frac{\partial^2 f}{\partial y \partial x}$  exist and are continuous at  $(x_0, y_0)$ , then*

$$\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$$

**Proof:** Here is an outline of the proof. The details are given as footnotes at the end of the outline. Fix  $x_0$  and  $y_0$  and define<sup>(1)</sup>

$$F(h, k) = \frac{1}{hk} [f(x_0 + h, y_0 + k) - f(x_0, y_0 + k) - f(x_0 + h, y_0) + f(x_0, y_0)]$$

Then, by the mean value theorem

$$\begin{aligned} F(h, k) &\stackrel{2}{=} \frac{1}{h} \left[ \frac{\partial f}{\partial y}(x_0 + h, y_0 + \theta_1 k) - \frac{\partial f}{\partial y}(x_0, y_0 + \theta_1 k) \right] \\ &\stackrel{3}{=} \frac{\partial}{\partial x} \frac{\partial f}{\partial y}(x_0 + \theta_2 h, y_0 + \theta_1 k) \\ F(h, k) &\stackrel{4}{=} \frac{1}{k} \left[ \frac{\partial f}{\partial x}(x_0 + \theta_3 h, y_0 + k) - \frac{\partial f}{\partial x}(x_0 + \theta_3 h, y_0) \right] \\ &\stackrel{5}{=} \frac{\partial}{\partial y} \frac{\partial f}{\partial x}(x_0 + \theta_3 h, y_0 + \theta_4 k) \end{aligned}$$

for some  $0 < \theta_1, \theta_2, \theta_3, \theta_4 < 1$ . All of  $\theta_1, \theta_2, \theta_3, \theta_4$  depend on  $x_0, y_0, h, k$ . Hence

$$\frac{\partial}{\partial x} \frac{\partial f}{\partial y}(x_0 + \theta_2 h, y_0 + \theta_1 k) = \frac{\partial}{\partial y} \frac{\partial f}{\partial x}(x_0 + \theta_3 h, y_0 + \theta_4 k)$$

for all  $h$  and  $k$ . Taking the limit  $(h, k) \rightarrow (0, 0)$  and using the assumed continuity of both partial derivatives at  $(x_0, y_0)$  gives

$$\frac{\partial}{\partial x} \frac{\partial f}{\partial y}(x_0, y_0) = \frac{\partial}{\partial y} \frac{\partial f}{\partial x}(x_0, y_0)$$

## The Details

- (1) We define  $F(h, k)$  in this way because both partial derivatives  $\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$  and  $\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0)$  are defined as limits of  $F(h, k)$  as  $h, k \rightarrow 0$ . For example,

$$\begin{aligned} \lim_{h \rightarrow 0} F(h, k) &= \frac{1}{k} \left[ \frac{\partial f}{\partial x}(x_0, y_0 + k) - \frac{\partial f}{\partial x}(x_0, y_0) \right] \\ \implies \lim_{k \rightarrow 0} \lim_{h \rightarrow 0} F(h, k) &= \lim_{k \rightarrow 0} \frac{1}{k} \left[ \frac{\partial f}{\partial x}(x_0, y_0 + k) - \frac{\partial f}{\partial x}(x_0, y_0) \right] \\ &= \frac{\partial}{\partial y} \frac{\partial f}{\partial x}(x_0, y_0) \end{aligned}$$

Similarly,

$$\begin{aligned}\lim_{k \rightarrow 0} F(h, k) &= \frac{1}{h} \left[ \frac{\partial f}{\partial y}(x_0 + h, y_0) - \frac{\partial f}{\partial y}(x_0, y_0) \right] \\ \implies \lim_{h \rightarrow 0} \lim_{k \rightarrow 0} F(h, k) &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\partial f}{\partial y}(x_0 + h, y_0) - \frac{\partial f}{\partial y}(x_0, y_0) \right] \\ &= \frac{\partial}{\partial x} \frac{\partial f}{\partial y}(x_0, y_0)\end{aligned}$$

(2) Define  $G(y) = f(x_0 + h, y) - f(x_0, y)$ . By the mean value theorem

$$\begin{aligned}F(h, k) &= \frac{1}{h} \left[ \frac{G(y_0+k) - G(y_0)}{k} \right] \\ &= \frac{1}{h} \frac{dG}{dy}(y_0 + \theta_1 k) \quad \text{for some } 0 < \theta_1 < 1 \\ &= \frac{1}{h} \left[ \frac{\partial f}{\partial y}(x_0 + h, y_0 + \theta_1 k) - \frac{\partial f}{\partial y}(x_0, y_0 + \theta_1 k) \right]\end{aligned}$$

(3) Define  $H(x) = \frac{\partial f}{\partial y}(x, y_0 + \theta_1 k)$ . By the mean value theorem

$$\begin{aligned}F(h, k) &= \frac{1}{h} \left[ H(x_0 + h) - H(x_0) \right] \\ &= \frac{dH}{dx}(x_0 + \theta_2 h) \quad \text{for some } 0 < \theta_2 < 1 \\ &= \frac{\partial}{\partial x} \frac{\partial f}{\partial y}(x_0 + \theta_2 h, y_0 + \theta_1 k)\end{aligned}$$

(4) Define  $A(x) = f(x, y_0 + k) - f(x, y_0)$ . By the mean value theorem

$$\begin{aligned}F(h, k) &= \frac{1}{k} \left[ \frac{A(x_0+h) - A(x_0)}{h} \right] \\ &= \frac{1}{k} \frac{dA}{dx}(x_0 + \theta_3 h) \quad \text{for some } 0 < \theta_3 < 1 \\ &= \frac{1}{k} \left[ \frac{\partial f}{\partial x}(x_0 + \theta_3 h, y_0 + k) - \frac{\partial f}{\partial x}(x_0 + \theta_3 h, y_0) \right]\end{aligned}$$

(5) Define  $B(y) = \frac{\partial f}{\partial x}(x_0 + \theta_3 h, y)$ . By the mean value theorem

$$\begin{aligned}F(h, k) &= \frac{1}{k} \left[ B(y_0 + k) - B(y_0) \right] \\ &= \frac{dB}{dy}(y_0 + \theta_4 k) \quad \text{for some } 0 < \theta_4 < 1 \\ &= \frac{\partial}{\partial y} \frac{\partial f}{\partial x}(x_0 + \theta_3 h, y_0 + \theta_4 k)\end{aligned}$$

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