MATHEMATICS 226 December, 2011 Final Exam Solutions

1. Consider the ellipsoid

$$x^2 + \tfrac{y^2}{4} + \tfrac{z^2}{9} = 21$$

(a) Find an equation for the tangent plane at the point (a, b, c) on the ellpsoid.
(b) At which points on the ellipsoid is the tangent plane parallel to the plane $x + 4y + \frac{4}{3}z = 471$?
(c) For which points on the ellipsoid does the tangent plane pass through $(7, 0, 0)$? Describe this set geometrically.

Solution. (a) The vector $\boldsymbol{\nabla}\big(x^2 + \frac{y^2}{4} + \frac{z^2}{9}\big)\big|_{(x,y,z)=(a,b,c)} = \big\langle 2a, \frac{1}{2}b, \frac{2}{9}c\big\rangle$ is normal to the ellpsoid at (a, b, c). Hence

$$\big\langle 2a, \tfrac{1}{2}b, \tfrac{2}{9}c\big\rangle \cdot \langle x-a, y-b, z-c\rangle = 0 \quad \text{or} \quad \boxed{2ax + \tfrac{1}{2}by + \tfrac{2}{9}cz = 2a^2 + \tfrac{b^2}{2} + \tfrac{2}{9}c^2 = 42}$$

is the equation of the tangent plane. (Recall that (a, b, c) is on the ellipsoid and so obeys $a^2 + \frac{b^2}{4} + \frac{c^2}{9} = 21$.)
(b) For the tangent plane to be parallel to $x + 4y + \frac{4}{3}z = 471$ we need

$$\big\langle 2a, \tfrac{1}{2}b, \tfrac{2}{9}c\big\rangle \parallel \big\langle 1, 4, \tfrac{4}{3}\big\rangle \iff \big\langle 2a, \tfrac{1}{2}b, \tfrac{2}{9}c\big\rangle = t\,\big\langle 1, 4, \tfrac{4}{3}\big\rangle \iff t = 2a,\ \tfrac{1}{2}b = 4t = 8a,\ \tfrac{2}{9}c = \tfrac{4}{3}t = \tfrac{8}{3}a$$

So we need $b = 16a$ and $c = 12a$. As (a, b, c) must also be on the ellpsoid, we need

$$a^2 + \tfrac{(16a)^2}{4} + \tfrac{(12a)^2}{9} = 21 \iff (1 + 64 + 16)a^2 = 21 \iff a^2 = \tfrac{21}{81} = \iff a = \pm\tfrac{\sqrt{21}}{9}$$
$$\iff \boxed{(a, b, c) = \pm\tfrac{\sqrt{21}}{9}(1, 16, 12)}$$

(c) The tangent plane passes through $(7, 0, 0)$ when $(x, y, z) = (7, 0, 0)$ obeys $2ax + \frac{1}{2}by + \frac{2}{9}cz = 42$ which is the case when

$$14a = 42 \iff a = 3$$

That is the set of points is the intersection of the ellpsoid with the plane $x = 3$ and so is an ellipse. The ellipse is

$$\boxed{x = 3 \qquad \tfrac{y^2}{4} + \tfrac{z^2}{9} = 21 - 9 = 12}$$

The ellipse has semiminor axis of length $4\sqrt{3}$ and semimajor axis of length $6\sqrt{3}$.

2. A gas is known to satisfy the van der Waals law $p = \frac{2T}{V-1} - \frac{4}{V^2}$ where p =pressure, V =volume and T =temperature. This equation of state is used to determine V as a function of p and T near $p = T = 1$. In particular $V(1, 1) = 2$.

(a) Determine $\frac{\partial V}{\partial p}(1, 1)$, $\frac{\partial V}{\partial T}(1, 1)$ and $\frac{\partial^2 V}{\partial p^2}(1, 1)$.
(b) Measurements are made and it is found that $p = 1 \pm 0.02$ and $T = 1 \pm 0.02$. Find the approximate maximum error in the value of V if we take $V = 2$.

Solution. (a) Viewing V as a function of p and T and applying $\frac{\partial}{\partial p}$ and $\frac{\partial}{\partial T}$ and then evaluating at $p = T = 1,\ V = 2$ yields

$$1 = -\tfrac{2T}{(V-1)^2}\tfrac{\partial V}{\partial p} + \tfrac{8}{V^3}\tfrac{\partial V}{\partial p} \Rightarrow 1 = (-2+1)\tfrac{\partial V}{\partial p}(1, 1) \Rightarrow \boxed{\tfrac{\partial V}{\partial p}(1, 1) = -1}$$
$$0 = \tfrac{2}{V-1} - \tfrac{2T}{(V-1)^2}\tfrac{\partial V}{\partial T} + \tfrac{8}{V^3}\tfrac{\partial V}{\partial T} \Rightarrow (-2+1)\tfrac{\partial V}{\partial T}(1, 1) = -2 \Rightarrow \boxed{\tfrac{\partial V}{\partial T}(1, 1) = 2}$$

Differentiating the first equation with respect to p and then evaluating at $p = T = 1,\ V = 2$ yields

$$0 = \big[-\tfrac{2T}{(V-1)^2} + \tfrac{8}{V^3}\big]\tfrac{\partial^2 V}{\partial p^2} + \big[\tfrac{4T}{(V-1)^3} - \tfrac{24}{V^4}\big]\big(\tfrac{\partial V}{\partial p}\big)^2$$
$$\Rightarrow 0 = \big[-2+1\big]\tfrac{\partial^2 V}{\partial p^2} + \big[4 - \tfrac{3}{2}\big]\big(-1\big)^2 \Rightarrow \boxed{\tfrac{\partial^2 V}{\partial p^2}(1, 1) = \tfrac{5}{2}}$$

(b) The approximate maximum error is

$$|V(p,T) - V(1,1)| \approx \left|\frac{\partial V}{\partial p}(1,1)\Delta p + \frac{\partial V}{\partial T}(1,1)\Delta T\right| = |-1(\pm.02) + 2(\pm.02)| = \boxed{0.06}$$

3. Consider the function

$$f(x,y) = 2x^3 - 6xy + y^2 + 4y$$

(a) Find and classify all of the critical points of $f(x,y)$.
(b) Find the maximum and minimum values of $f(x,y)$ in the triangle with vertices $(1,0)$, $(0,1)$ and $(1,1)$.

Solution. (a)

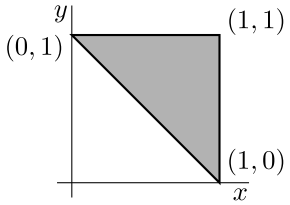
$$\begin{aligned} f &= 2x^3 - 6xy + y^2 + 4y \\ f_x &= 6x^2 - 6y \qquad & f_{xx} &= 12x \quad f_{xy} = -6 \\ f_y &= -6x + 2y + 4 \qquad & f_{yy} &= 2 \end{aligned}$$

The critical points are the solutions of

$$\begin{aligned} & f_x = 0 \quad f_y = 0 \\ \iff\ & y = x^2 \quad y - 3x + 2 = 0 \\ \iff\ & y = x^2 \quad x^2 - 3x + 2 = 0 \\ \iff\ & y = x^2 \quad x = 1 \text{ or } 2 \end{aligned}$$

So, there are two critical points: $(1,1)$, $(2,4)$.

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(1,1)$	$12 \times 2 - (-6)^2 < 0$		saddle point
$(2,4)$	$24 \times 2 - (-6)^2 > 0$	24	local min



(b) There are no critical points in the interior of the allowed region, so both the maximum and the minimum occur only on the boundary. The boundary consists of the line segments (i) $x = 1$, $0 \le y \le 1$, (ii) $y = 1$, $0 \le x \le 1$ and (iii) $y = 1 - x$, $0 \le x \le 1$.

First, we look at the part of the boundary with $x = 1$. There $f = y^2 - 2y + 2$. As $\frac{d}{dy}(y^2 - 2y + 2) = 2y - 2$ vanishes only at $y = 1$, the max and min of $y^2 - 2y + 2$ for $0 \le y \le 1$ must occur either at $y = 0$, where $f = 2$, or at $y = 1$, where $f = 1$.

Next, we look at the part of the boundary with $y = 1$. There $f = 2x^3 - 6x + 5$. As $\frac{d}{dx}(2x^3 - 6x + 5) = 6x^2 - 6$, the max and min of $2x^3 - 6x + 5$ for $0 \le x \le 1$ must occur either at $x = 0$, where $f = 5$, or at $x = 1$, where $f = 1$.

Next, we look at the part of the boundary with $y = 1 - x$. There $f = 2x^3 - 6x(1-x) + (1-x)^2 + 4(1-x) = 2x^3 + 7x^2 - 12x + 5$. As $\frac{d}{dx}(2x^3 + 7x^2 - 12x + 5) = 6x^2 + 14x - 12 = 2(3x - 2)(x + 3)$, the max and min of $2x^3 + 7x^2 - 12x + 5$ for $0 \le x \le 1$ must occur either at $x = 0$, where $f = 5$, or at $x = 1$, where $f = 2$, or at $x = \frac{2}{3}$, where $f = 2(\frac{8}{27}) - 6(\frac{2}{3})(\frac{1}{3}) + \frac{1}{9} + \frac{4}{3} = \frac{16-36+3+36}{27} = \frac{19}{27}$.

All together, we have the following candidates for max and min

point	$(1,0)$	$(1,1)$	$(0,1)$	$\left(\frac{2}{3}, \frac{1}{3}\right)$
value of f	2	1	5	$\frac{19}{27}$

The largest and smallest values of f in this table are $\boxed{\text{min}= \frac{19}{27},\ \text{max}= 5}$.

4. Find the points on the ellipse $2x^2 + 4xy + 5y^2 = 30$ which are closest to and farthest from the origin.

   Solution. Let (x, y) be a point on $2x^2 + 4xy + 5y^2 = 30$. We wish to maximize and minimize $x^2 + y^2$ subject to $2x^2 + 4xy + 5y^2 = 30$. Define $L(x, y, \lambda) = x^2 + y^2 - \lambda(2x^2 + 4xy + 5y^2 - 30)$. Then

$$\begin{array}{lcll} 0 = L_x = 2x - \lambda(4x + 4y) & \implies & (1 - 2\lambda)x - 2\lambda y \;= 0 & (1) \\ 0 = L_y = 2y - \lambda(4x + 10y) & \implies & -2\lambda x + (1 - 5\lambda)y = 0 & (2) \\ 0 = L_\lambda = 2x^2 + 4xy + 5y^2 - 30 & & & \end{array}$$

   Note that λ cannot be zero because if it is, (1) forces $x = 0$ and (2) forces $y = 0$, but $(0, 0)$ is not on the ellipse. So equation (1) gives $y = \frac{1-2\lambda}{2\lambda}x$. Subbing this into equation (2) gives $-2\lambda x + \frac{(1-5\lambda)(1-2\lambda)}{2\lambda}x = 0$. To get a nonzero (x, y) we need

$$-2\lambda + \tfrac{(1-5\lambda)(1-2\lambda)}{2\lambda} = 0 \iff 0 = -4\lambda^2 + (1 - 5\lambda)(1 - 2\lambda) = 6\lambda^2 - 7\lambda + 1 = (6\lambda - 1)(\lambda - 1)$$

   So λ must be either 1 or $\frac{1}{6}$. Subbing these into either (1) or (2) gives

$$\begin{array}{lclclclcl} \lambda = 1 & \implies & -x - 2y = 0 & \implies & x = -2y & \implies & 8y^2 - 8y^2 + 5y^2 \;= 30 & \implies & y = \pm\sqrt{6} \\ \lambda = \frac{1}{6} & \implies & \frac{2}{3}x - \frac{1}{3}y = 0 & \implies & y = 2x & \implies & 2x^2 + 8x^2 + 20x^2 = 30 & \implies & x = \pm 1 \end{array}$$

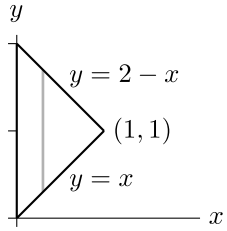
   The farthest points are $\boxed{\pm\sqrt{6}(-2, 1)}$. The nearest points are $\boxed{\pm(1, 2)}$.

5. Reverse the order of integration of

$$\int_0^1 \left(\int_0^y \frac{e^x}{1 - x} dx \right) dy + \int_1^2 \left(\int_0^{2-y} \frac{e^x}{1 - x} dx \right) dy$$

   and then evaluate the integral.

   Solution. The domain of integration is sketched below.



   So the integral is

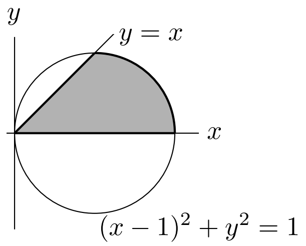
$$\int_0^1 dx \int_x^{2-x} dy \; \tfrac{e^x}{1-x} = \int_0^1 dx \; \tfrac{e^x}{1-x}(2 - x - x) = 2\int_0^1 dx \; e^x = 2\big[e^x\big]_0^1 = \boxed{2(e - 1)}$$

6. Consider the integral

$$\int_0^1 dy \int_y^{1+\sqrt{1-y^2}} dx \; \sqrt{x}$$

   (a) Sketch the region of integration.
   (b) Express the integral in polar coordinates.
   (c) Evaluate it.

   Solution. The domain of integration is $0 \le y \le 1$, $y \le x \le 1 + \sqrt{1 - y^2}$. The boundary $x = 1 + \sqrt{1 - y^2}$ is the circle $(x - 1)^2 + y^2 = 1$. The domain is sketched below.



In polar coordinates, the line $y = x$ is $\theta = \frac{\pi}{4}$ and the circle $x^2 - 2x + y^2 = 0$ is $r = 2\cos\theta$. So

$$\begin{aligned}
\int_0^1 dy \int_y^{1+\sqrt{1-y^2}} dx\ \sqrt{x} &= \boxed{\textstyle\int_0^{\pi/4} d\theta \int_0^{2\cos\theta} dr\ r\sqrt{r\cos\theta}} \\
&= \int_0^{\pi/4} d\theta \int_0^{2\cos\theta} dr\ r^{3/2}\sqrt{\cos\theta} \\
&= \int_0^{\pi/4} d\theta\ \tfrac{2}{5}(2\cos\theta)^{5/2}\sqrt{\cos\theta} \\
&= \tfrac{8\sqrt{2}}{5} \int_0^{\pi/4} d\theta\ \cos^3\theta = \tfrac{8\sqrt{2}}{5} \int_0^{\pi/4} d\theta\ (1 - \sin^2\theta)\cos\theta \\
&= \tfrac{8\sqrt{2}}{5} \Big[\sin\theta - \tfrac{1}{3}\sin^3\theta\Big]_0^{\pi/4} = \tfrac{8\sqrt{2}}{5}\Big[\tfrac{1}{\sqrt{2}} - \tfrac{1}{6}\tfrac{1}{\sqrt{2}}\Big] = \boxed{\tfrac{4}{3}}
\end{aligned}$$

7. A cylindrical hole of radius a cm is drilled through the centre of a solid sphere of radius b cm (where $b > a$) and density $\rho\, y^2$ gm/cm^3. Find the mass of the *remaining part* of the sphere.

Solution. In cylindrical coordinates, the equation of the sphere is $r^2 + z^2 = b^2$, the equation of the boundary of the hole is $r = a$, the density is $\rho\, r^2 \sin^2\theta$ and the volume element is $dV = r dr d\theta dz$. So the mass is

$$\begin{aligned}
M &= 2\int_0^{2\pi} d\theta \int_a^b dr\ r \int_0^{\sqrt{b^2-r^2}} dz\ \rho r^2 \sin^2\theta \\
&= 2\rho \int_0^{2\pi} d\theta \int_a^b dr\ r^3 \sqrt{b^2 - r^2}\,\sin^2\theta
\end{aligned}$$

To evaluate the r integral, make the change of variables $t = b^2 - r^2$, $dt = -2r\,dr$. Then

$$\begin{aligned}
M &= -\rho \int_0^{2\pi} d\theta \int_{b^2-a^2}^{0} dt\ (b^2 - t)\sqrt{t}\sin^2\theta \\
&= -\rho \int_0^{2\pi} d\theta\ \sin^2\theta\ \big[\tfrac{2}{3}b^2 t^{3/2} - \tfrac{2}{5}t^{5/2}\big]_{b^2-a^2}^{0} \\
&= \rho \int_0^{2\pi} d\theta\ \sin^2\theta\ \big[\tfrac{2}{3}b^2(b^2-a^2)^{3/2} - \tfrac{2}{5}(b^2-a^2)^{5/2}\big]
\end{aligned}$$

To do the θ integral, we use either

$$\int_0^{2\pi} d\theta\ \sin^2\theta = \tfrac{1}{2}\int_0^{2\pi} d\theta\ \big(1 - \cos(2\theta)\big) = \tfrac{1}{2}\big[\theta + \tfrac{1}{2}\sin(2\theta)\big]_0^{2\pi} = \pi$$

or

$$\int_0^{2\pi} d\theta\ \sin^2\theta = \int_0^{2\pi} d\theta\ \cos^2\theta \quad\Longrightarrow\quad \int_0^{2\pi} d\theta\ \sin^2\theta = \tfrac{1}{2}\int_0^{2\pi} d\theta\ [\sin^2\theta + \cos^2\theta] = \tfrac{1}{2}\int_0^{2\pi} d\theta\ 1 = \pi$$

So

$$\begin{aligned}
M &= \rho\big[\tfrac{2}{3}b^2(b^2-a^2)^{3/2} - \tfrac{2}{5}(b^2-a^2)^{5/2}\big]\pi \\
&= 2\rho\big[\tfrac{1}{3}b^2 - \tfrac{1}{5}(b^2-a^2)\big](b^2-a^2)^{3/2}\pi \\
&= \boxed{2\rho\big[\tfrac{2}{15}b^2 + \tfrac{1}{5}a^2\big](b^2-a^2)^{3/2}\pi}
\end{aligned}$$

8. Consider the function

$$f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(a) Evaluate, if possible, $\frac{\partial f}{\partial x}(0,0)$ and $\frac{\partial f}{\partial y}(0,0)$.
(b) Is $f(x,y)$ differentiable at $(0,0)$? You must justify your answer to receive any marks.

Solution. (a) *Solution 1*: By definition

$$\frac{\partial f}{\partial x}(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{(\Delta x)^3}{(\Delta x)^2+0^2} - 0}{\Delta x} = \lim_{\Delta x \to 0} 1 = \boxed{1}$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{\Delta y \to 0} \frac{f(0, \Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\frac{0^3}{0+(\Delta y)^2} - 0}{\Delta y} = \lim_{\Delta y \to 0} 0 = \boxed{0}$$

(a) *Solution 2:* Observe that $f(x,0) = x$ for all x (both $(x,y) \neq (0,0)$ and $(x,y) = (0,0)$). Hence $\frac{\partial f}{\partial x}(0,0) = \frac{d}{dx}x\big|_{x=0} = 1$. Similarly, $f(0,y) = 0$ for all y implies $\frac{\partial f}{\partial y}(0,0) = \frac{d}{dy}0\big|_{y=0} = 0$.
(b) By definition, $f(x,y)$ differentiable at $(0,0)$ if and only if

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - f_x(0,0)x - f_y(0,0)y}{\sqrt{x^2+y^2}} = 0$$

For all $(x,y) \neq (0,0)$

$$\frac{f(x,y) - f_x(0,0)x - f_y(0,0)y}{\sqrt{x^2+y^2}} = \frac{\frac{x^3}{x^2+y^2} - x}{\sqrt{x^2+y^2}} = \frac{x^3 - x(x^2+y^2)}{[x^2+y^2]^{3/2}} = \frac{-xy^2}{[x^2+y^2]^{3/2}}$$

Switching to polar coordinates

$$\frac{-xy^2}{[x^2+y^2]^{3/2}} = \frac{-(r\cos\theta)(r\sin\theta)^2}{r^3} = -\cos\theta\sin^2\theta$$

This does not have a limit as $(x,y) \to (0,0)$. (For example, when $\theta = \frac{\pi}{4}$, this is $-2^{-3/2}$ for all r and when $r = 0$, this is 0 for all r.) So $f(x,y)$ is $\boxed{\text{not differentiable}}$ at $(0,0)$.