Limits

Notation.
- $\mathbb{N}$ is the set $\{1, 2, 3, \cdots\}$ of all natural numbers
- $\mathbb{R}$ is the set of all real numbers
- $\forall$ is read “for all”
- $\exists$ is read “there exists”
- $\in$ is read “element of”
- $\notin$ is read “not an element of”
- $\{\ A \mid B \}$ is read “the set of all $A$ such that $B$”
- If $S$ is a set and $T$ is a subset of $S$, then $S \setminus T$ is $\{ x \in S \mid x \notin T \}$, the set $S$ with the elements of $T$ removed.
- if $n$ is a natural number, $\mathbb{R}^n$ is used for both the set of $n$–component vectors $\langle x_1, x_2, \cdots, x_n \rangle$ and the set of points $(x_1, x_2, \cdots, x_n)$ with $n$–coordinates.
- If $S$ and $T$ are sets, then $f : S \to T$ means that $f$ is a function which assigns to each element of $S$ an element of $T$.
- $[a, b] = \{ x \in \mathbb{R} \mid a \leq x \leq b \}$
- $(a, b) = \{ x \in \mathbb{R} \mid a < x \leq b \}$
- $[a, b) = \{ x \in \mathbb{R} \mid a \leq x < b \}$
- $(a, b) = \{ x \in \mathbb{R} \mid a < x < b \}$

Roughly speaking, $\lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x}) = \vec{L}$ means that $\vec{f}(\vec{x})$ approaches $\vec{L}$ as $\vec{x}$ approaches $\vec{a}$. Here is the precise definition of limit, and a couple of related definitions.

Definition 1 Let $m, n \in \mathbb{N}$.

(a) Let $\vec{a} \in \mathbb{R}^n$ and $\vec{L} \in \mathbb{R}^m$, and let $\vec{f} : \mathbb{R}^n \setminus \{\vec{a}\} \to \mathbb{R}^m$. Then $\lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x}) = \vec{L}$ if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \text{such that} \ \left| \vec{f}(\vec{x}) - \vec{L} \right| < \varepsilon \ \text{whenever} \ 0 < |\vec{x} - \vec{a}| < \delta$$

(b) Let $\vec{f} : \mathbb{R}^n \to \mathbb{R}^m$. Then $f$ is continuous at $\vec{a} \in \mathbb{R}^n$ if $\lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x}) = \vec{f}(\vec{a})$ and $\vec{f}$ is continuous on $\mathbb{R}^n$ if it is continuous at every $\vec{a} \in \mathbb{R}^n$.

Remark 2

(a) Here is what that definition of limit says. Suppose you have a magic microscope whose magnification can be set as high as you like. Suppose that when the magnification is set to $\frac{1}{\varepsilon}$, you can only see those points whose distance from $\vec{L}$ is less than $\varepsilon$. The definition
says that no matter how high you set the magnification, (i.e. no matter how small you set $\varepsilon > 0$), you will be able to see $\vec{f}(\vec{x})$ whenever $\vec{x}$ is close enough to $\vec{a}$ (if the distance from $\vec{x}$ to $\vec{a}$ is less than $\delta$, then you will certainly see $\vec{f}(\vec{x})$).

(b) Definition 1.a, of $\lim_{\vec{x} \to \vec{a}} \vec{f}(\vec{x})$, is set up so that the function $\vec{f}(\vec{x})$ is never evaluated at $\vec{x} = \vec{a}$. Indeed $\vec{f}(\vec{x})$ need not even be defined at $\vec{x} = \vec{a}$. This is exactly what happens in the definition of the derivative $h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a}$. (In this case $f(x) = \frac{h(x) - h(a)}{x - a}$.)

We’ll first do a couple of examples with $m = n = 1$. We’ll do higher dimensional examples later.

**Example 3** In Example 2 of the notes “A Little Logic” we saw that the statement

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that } \ |x| < \delta \text{ then } x^2 < \varepsilon$$

is true. Consequently

$$\lim_{x \to 0} x^2 = 0$$

**Example 4** In this example, we consider $\lim_{x \to 0} \sin \frac{1}{x}$. So fix any real number $L$ and let

- $S(\delta, \varepsilon)$ be the statement “$|\sin \frac{1}{x} - L| < \varepsilon$ whenever $0 < |x| < \delta$”
- $T(\varepsilon)$ be the statement “$\exists \delta > 0$ such that $S(\delta, \varepsilon)$” or $\exists \delta > 0$ such that $|\sin \frac{1}{x} - L| < \varepsilon$ whenever $0 < |x| < \delta$
- $U$ be the statement “$\forall \varepsilon > 0 \ T(\varepsilon)$” or $\forall \varepsilon > 0 \ \exists \delta > 0$ such that $|\sin \frac{1}{x} - L| < \varepsilon$ whenever $0 < |x| < \delta$

Then

- Fix any $\varepsilon > 0$ and any $\delta > 0$. The statement $S(\delta, \varepsilon)$ is true if all values of $\sin \frac{1}{x}$, with $0 < |x| < \delta$, lie in the interval $(L - \varepsilon, L + \varepsilon)$. As $x$ runs over the interval $(0, \delta)$, (so that, in particular, $0 < |x| < \delta$) $\frac{1}{x}$ covers the set $\left(\frac{1}{\delta}, \infty\right)$. This contains many intervals of length $2\pi$ and hence many periods of sin. So, as $x$ runs over the interval $(0, \delta)$, $\sin \frac{1}{x}$ covers all of $[-1, 1]$. So $S(\delta, \varepsilon)$ is true if and only if the interval $[-1, 1]$ is contained in the interval $(L - \varepsilon, L + \varepsilon)$. In particular, when $\varepsilon < 1$, the interval $(L - \varepsilon, L + \varepsilon)$, which has length $2\varepsilon$, is shorter than $[-1, 1]$ and cannot contain it, so that $S(\delta, \varepsilon)$ is false.
- Because $S(\delta, \varepsilon)$ is false for all $\delta > 0$ when $\varepsilon < 1$, $T(\varepsilon)$ is false for all $\varepsilon < 1$.
- $U$ is false since, as we have just seen, $T(\varepsilon)$ is false for at least one $\varepsilon > 0$. For example $T\left(\frac{1}{\pi}\right)$ is false.

In conclusion, $\sin \frac{1}{x}$ has no limit as $x \to 0$.  

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September 24, 2011  

Limits 2
Theorem 5  Let \( n \in \mathbb{N}, \vec{a}, \vec{b} \in \mathbb{R}^n, F, G \in \mathbb{R} \) and

\[
f, g : \mathbb{R}^n \setminus \{\vec{a}\} \to \mathbb{R} \quad \vec{X} : \mathbb{R}^n \setminus \{\vec{b}\} \to \mathbb{R}^n \setminus \{\vec{a}\} \quad \gamma : \mathbb{R} \to \mathbb{R}
\]

Assume that

\[
\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = F \quad \lim_{\vec{x} \to \vec{a}} g(\vec{x}) = G \quad \lim_{\vec{y} \to \vec{b}} \vec{X}(\vec{y}) = \vec{a} \quad \lim_{t \to F} \gamma(t) = \gamma(F) = \Gamma
\]

Then

(a) \[ \lim_{\vec{x} \to \vec{a}} [f(\vec{x}) + g(\vec{x})] = F + G \]

(b) \[ \lim_{\vec{x} \to \vec{a}} f(\vec{x})g(\vec{x}) = FG \]

(c) \[ \lim_{\vec{x} \to \vec{a}} \frac{f(\vec{x})}{g(\vec{x})} = \frac{F}{G} \quad \text{if } G \neq 0 \]

(d) \[ \lim_{\vec{y} \to \vec{b}} \vec{X}(\vec{y}) = F \]

(e) \[ \lim_{\vec{x} \to \vec{a}} \gamma(f(\vec{x})) = \Gamma \]

Proof: Note that the \( \varepsilon \) and \( \delta \) in “\( \forall \varepsilon > 0 \ \exists \delta > 0 \) such that \( S(\delta, \varepsilon) \)” are dummy variables, just as \( x \) is a dummy variable in \( \int_0^1 x \, dx \). You may replace \( \varepsilon \) and \( \delta \) by whatever symbols you like. The hypotheses of this theorem say that

\[
\forall \varepsilon_f > 0 \ \exists \delta_f > 0 \text{ such that } |f(\vec{x}) - F| < \varepsilon_f \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta_f \quad (1)
\]

\[
\forall \varepsilon_g > 0 \ \exists \delta_g > 0 \text{ such that } |g(\vec{x}) - G| < \varepsilon_g \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta_g \quad (2)
\]

\[
\forall \varepsilon_X > 0 \ \exists \delta_X > 0 \text{ such that } |\vec{X}(\vec{y}) - \vec{a}| < \varepsilon_X \text{ whenever } 0 < |\vec{y} - \vec{b}| < \delta_X \quad (3)
\]

\[
\forall \varepsilon_{\gamma} > 0 \ \exists \delta_{\gamma} > 0 \text{ such that } |\gamma(t) - \Gamma| < \varepsilon_{\gamma} \text{ whenever } 0 < |t - F| < \delta_{\gamma} \quad (4)
\]

(a) We are to prove that

\[
\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that } |f(\vec{x}) + g(\vec{x}) - F - G| < \varepsilon \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta
\]

So pick any \( \varepsilon > 0 \). We must prove that there is a \( \delta > 0 \) such that

\[
|f(\vec{x}) + g(\vec{x}) - F - G| < \varepsilon \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta
\]

Observe that

\[
|f(\vec{x}) + g(\vec{x}) - F - G| = |[f(\vec{x}) - F] + [g(\vec{x}) - G]| \leq |f(\vec{x}) - F| + |g(\vec{x}) - G|
\]
Set $\varepsilon_1 = \frac{\varepsilon}{2}$ and $\varepsilon_2 = \frac{\varepsilon}{2}$. By (1) with $\varepsilon_f = \varepsilon_1$ and (2) with $\varepsilon_g = \varepsilon_2$,

\[
\exists \delta_1 > 0 \text{ such that } |f(\vec{x}) - F| < \varepsilon_1 \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta_1 \\
\exists \delta_2 > 0 \text{ such that } |g(\vec{x}) - G| < \varepsilon_2 \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta_2
\]

Choose $\delta = \min \{\delta_1, \delta_2\}$. Then whenever $0 < |\vec{x} - \vec{a}| < \delta$ we also have $0 < |\vec{x} - \vec{a}| < \delta_1$ and $0 < |\vec{x} - \vec{a}| < \delta_2$ so that

\[
|f(\vec{x}) + g(\vec{x}) - F - G| \leq |f(\vec{x}) - F| + |g(\vec{x}) - G| < \varepsilon_1 + \varepsilon_2 = \varepsilon
\]

(b) is a homework assignment.

(c) We are to prove that

\[
\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that } \left| \frac{f(\vec{x})}{g(\vec{x})} - \frac{F}{G} \right| < \varepsilon \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta
\]

So pick any $\varepsilon > 0$. We must prove that there is a $\delta > 0$ such that

\[
\left| \frac{f(\vec{x})}{g(\vec{x})} - \frac{F}{G} \right| < \varepsilon \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta
\]

Set $\varepsilon_1 = \frac{1}{6}|G|\varepsilon$ and $\varepsilon_2 = \frac{G^2}{6(|F|+1)}\varepsilon$. By (1) with $\varepsilon_f = \varepsilon_1$, (2) with $\varepsilon_g = \varepsilon_2$ and (2) with $\varepsilon_g = \frac{1}{2}|G|$, we also have $0 < |\vec{x} - \vec{a}| < \delta_1$ and $0 < |\vec{x} - \vec{a}| < \delta_2$ and $0 < |\vec{x} - \vec{a}| < \delta_3$ so that

\[
\frac{|f(\vec{x})|}{|g(\vec{x})|} - \frac{F}{G} = \frac{|f(\vec{x})G - Fg(\vec{x})|}{|g(\vec{x})||G|} = \frac{|f(\vec{x}) - F||g(\vec{x}) - G|}{|g(\vec{x})||G|} \\
\leq \frac{|f(\vec{x}) - F||G| + |F||g(\vec{x}) - G|}{|g(\vec{x})||G|} \\
\leq \frac{\varepsilon_1 |G| + |F|\varepsilon_2}{\frac{1}{2}|G||G|} \text{ since } |g(\vec{x})| = |g(\vec{x}) - G + G| \geq |G| - |g(\vec{x}) - G| \geq \frac{1}{2}|G| \\
= \frac{1}{6}|G|\varepsilon + \frac{|F|}{G^2/2} \frac{G^2}{6(|F|+1)}\varepsilon = \frac{\varepsilon}{3} + \frac{1}{3} \frac{|F|}{|F|+1} \varepsilon \\
< \varepsilon
\]

(d) We are to prove that

\[
\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that } |f((\vec{x}) - F) - F| < \varepsilon \text{ whenever } 0 < |\vec{y} - \vec{b}| < \delta
\]
So pick any $\varepsilon > 0$. We must prove that there is a $\delta > 0$ such that

$$|f(\vec{X}(\vec{y})) - F| < \varepsilon \text{ whenever } 0 < |\vec{y} - \vec{b}| < \delta$$

By (1) with $\varepsilon_f = \varepsilon$

$$\exists \delta_f > 0 \text{ such that } |f(\vec{x}) - F| < \varepsilon \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta_f$$

and (3) with $\varepsilon_X = \delta_f$,

$$\exists \delta_X > 0 \text{ such that } |\vec{X}(\vec{y}) - \vec{a}| < \delta_f \text{ whenever } 0 < |\vec{y} - \vec{b}| < \delta_X$$

Choosing $\delta = \delta_X$, we have

$$0 < |\vec{y} - \vec{b}| < \delta = \delta_X \implies 0 < |\vec{X}(\vec{y}) - \vec{a}| < \delta_f \implies |f(\vec{X}(\vec{y})) - F| < \varepsilon$$

(e) has essentially the same proof as part (d). We are to prove that

$$\forall \varepsilon > 0 \ \exists \delta > 0 \text{ such that } |\gamma(f(\vec{x})) - \Gamma| < \varepsilon \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta$$

So pick any $\varepsilon > 0$. We must prove that there is a $\delta > 0$ such that

$$|\gamma(f(\vec{x})) - \Gamma| < \varepsilon \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta$$

By (4) with $\varepsilon_\gamma = \varepsilon$ and the hypothesis that $\gamma(F) = \Gamma$

$$\exists \delta_\gamma > 0 \text{ such that } |\gamma(t) - \Gamma| < \varepsilon \text{ whenever } |t - F| < \delta_\gamma$$

By (1) with $\varepsilon_f = \delta_\gamma$

$$\exists \delta_f > 0 \text{ such that } |f(\vec{x}) - F| < \delta_\gamma \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta_f$$

Choosing $\delta = \delta_f$, we have

$$0 < |\vec{x} - \vec{a}| < \delta = \delta_f \implies |f(\vec{x}) - F| < \delta_\gamma \implies |\gamma(f(\vec{x})) - \Gamma| < \varepsilon$$
Example 6  There is a typical application of Theorem 5. Here “\( \equiv \)" means that Theorem 5.a justifies that equality.

\[
\lim_{(x,y) \to (2,3)} (x + \sin y) = \lim_{(x,y) \to (2,3)} x + \lim_{(x,y) \to (2,3)} \sin y
\]

\[
= \lim_{(x,y) \to (2,3)} x + \sin \left( \lim_{(x,y) \to (2,3)} y \right)
\]

\[= 2 + \sin 3 \]

\[
\lim_{(x,y) \to (2,3)} (x^2 y^2 + 1) = \lim_{(x,y) \to (2,3)} x^2 y^2 + \lim_{(x,y) \to (2,3)} 1
\]

\[
= \left( \lim_{(x,y) \to (2,3)} x \right) \left( \lim_{(x,y) \to (2,3)} x \right) \left( \lim_{(x,y) \to (2,3)} y \right) \left( \lim_{(x,y) \to (2,3)} y \right) + 1
\]

\[= 2^2 3^2 + 1 \]

\[
\lim_{(x,y) \to (2,3)} \frac{x + \sin y}{x^2 y^2 + 1} = \frac{\lim_{(x,y) \to (2,3)} (x + \sin y)}{\lim_{(x,y) \to (2,3)} (x^2 y^2 + 1)}
\]

\[= \frac{2 + \sin 3}{3^7} \]

Here we have used that \( \sin x \) is a continuous function. In this course we shall assume that we already know that “standard single variable calculus functions” like \( \sin x \), \( \cos x \), \( e^x \) and so on are continuous.

Example 7  As a second example, we consider \( \lim_{(x,y) \to (0,0)} \frac{x^2 y}{x^2 + y^2} \). In this example, both the numerator, \( x^2 y \), and the denominator \( x^2 + y^2 \) tend to zero as \( (x, y) \) approaches \( (0,0) \), so we have to be more careful. A good way to see the behaviour of a function \( f(x, y) \) when \( (x, y) \) is close to \( (0,0) \) is to switch to the polar coordinates \( r, \theta \) using

\[
x = r \cos \theta \]

\[
y = r \sin \theta \]

Recall that the definition of \( \lim_{(x,y) \to (0,0)} f(x, y) = L \) is

\[
\forall \varepsilon > 0 \ \exists \delta > 0 \ such \ that \ |f(x, y) - L| < \varepsilon \ whenever \ 0 < |(x, y)| < \delta \quad (5)
\]

The condition \( 0 < |(x, y)| < \delta \) says that \( 0 < r < \delta \) and no restriction on \( \theta \). So substituting \( x = r \cos \theta \), \( y = r \sin \theta \) into (5) gives

\[
\forall \varepsilon > 0 \ \exists \delta > 0 \ such \ that \ |f(r \cos \theta, r \sin \theta) - L| < \varepsilon \ whenever \ 0 < r < \delta, \ 0 \leq \theta \leq 2\pi \ (6)
\]

For our current example

\[
\frac{x^2 y}{x^2 + y^2} = \frac{(r \cos \theta)^2 (r \sin \theta)}{r^2} = r \cos^2 \theta \sin \theta
\]

As \( |r \cos^2 \theta \sin \theta| \leq r \to 0 \) when \( r \to 0 \), we have

\[
\lim_{(x,y) \to (0,0)} \frac{x^2 y}{x^2 + y^2} = 0
\]
Example 8  As a third example, we consider \( \lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2} \). Once again, the best way to see the behaviour of \( \frac{x^2 - y^2}{x^2 + y^2} \) for \((x, y)\) close to \((0, 0)\) is to switch to polar coordinates.

\[
\frac{x^2 - y^2}{x^2 + y^2} = \frac{(r \cos \theta)^2 - (r \sin \theta)^2}{r^2} = \cos^2 \theta - \sin^2 \theta = \cos(2\theta)
\]

No matter how small you make \( \delta > 0 \), as \((x, y)\) runs over those points with \( r = |(x, y)| < \delta \), \( \frac{x^2 - y^2}{x^2 + y^2} \) takes all values in the interval \([-1, 1] \). So \( \lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2} \) does not exist.