

Limits

Notation.

- \mathbb{N} is the set $\{1, 2, 3, \dots\}$ of all natural numbers
- \mathbb{R} is the set of all real numbers
- \forall is read “for all”
- \exists is read “there exists”
- \in is read “element of”
- \notin is read “not an element of”
- $\{ A \mid B \}$ is read “the set of all A such that B ”
- If S is a set and T is a subset of S , then $S \setminus T$ is $\{ x \in S \mid x \notin T \}$, the set S with the elements of T removed.
- if n is a natural number, \mathbb{R}^n is used for both the set of n -component vectors $\langle x_1, x_2, \dots, x_n \rangle$ and the set of points (x_1, x_2, \dots, x_n) with n -coordinates.
- If S and T are sets, then $f : S \rightarrow T$ means that f is a function which assigns to each element of S an element of T .
- $[a, b] = \{ x \in \mathbb{R} \mid a \leq x \leq b \}$
 $(a, b) = \{ x \in \mathbb{R} \mid a < x < b \}$
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Roughly speaking, $\lim_{\vec{x} \rightarrow \vec{a}} \vec{f}(\vec{x}) = \vec{L}$ means that $\vec{f}(\vec{x})$ approaches \vec{L} as \vec{x} approaches \vec{a} . Here is the precise definition of limit, and a couple of related definitions.

Definition 1 Let $m, n \in \mathbb{N}$.

(a) Let $\vec{a} \in \mathbb{R}^n$ and $\vec{L} \in \mathbb{R}^m$, and let $\vec{f} : \mathbb{R}^n \setminus \{\vec{a}\} \rightarrow \mathbb{R}^m$. Then $\lim_{\vec{x} \rightarrow \vec{a}} \vec{f}(\vec{x}) = \vec{L}$ if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{such that} \quad |\vec{f}(\vec{x}) - \vec{L}| < \varepsilon \quad \text{whenever} \quad 0 < |\vec{x} - \vec{a}| < \delta$$

(b) Let $\vec{f} : \mathbb{R}^n \rightarrow \mathbb{R}^m$. Then f is continuous at $\vec{a} \in \mathbb{R}^n$ if $\lim_{\vec{x} \rightarrow \vec{a}} \vec{f}(\vec{x}) = \vec{f}(\vec{a})$ and \vec{f} is continuous on \mathbb{R}^n if it is continuous at every $\vec{a} \in \mathbb{R}^n$.

Remark 2

(a) Here is what that definition of limit says. Suppose you have a magic microscope whose magnification can be set as high as you like. Suppose that when the magnification is set to $\frac{1}{\varepsilon}$, you can only see those points whose distance from \vec{L} is less than ε . The definition

says that no matter how high you set the magnification, (i.e. no matter how small you set $\varepsilon > 0$), you will be able to see $\vec{f}(\vec{x})$ whenever \vec{x} is close enough to \vec{a} (if the distance from \vec{x} to \vec{a} is less than δ , then you will certainly see $\vec{f}(\vec{x})$).

(b) Definition 1.a, of $\lim_{\vec{x} \rightarrow \vec{a}} \vec{f}(\vec{x})$, is set up so that the function $\vec{f}(\vec{x})$ is never evaluated at $\vec{x} = \vec{a}$. Indeed $\vec{f}(\vec{x})$ need not even be defined at $\vec{x} = \vec{a}$. This is exactly what happens in the definition of the derivative $h'(a) = \lim_{x \rightarrow a} \frac{h(x) - h(a)}{x - a}$. (In this case $f(x) = \frac{h(x) - h(a)}{x - a}$.)

We'll first do a couple of examples with $m = n = 1$. We'll do higher dimensional examples later.

Example 3 In Example 2 of the notes "A Little Logic" we saw that the statement

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that if } |x| < \delta \text{ then } x^2 < \varepsilon$$

is true. Consequently

$$\lim_{x \rightarrow 0} x^2 = 0$$

Example 4 In this example, we consider $\lim_{x \rightarrow 0} \sin \frac{1}{x}$. So fix any real number L and let

- $S(\delta, \varepsilon)$ be the statement " $|\sin \frac{1}{x} - L| < \varepsilon$ whenever $0 < |x| < \delta$ ",
- $T(\varepsilon)$ be the statement " $\exists \delta > 0$ such that $S(\delta, \varepsilon)$ " or

$$\exists \delta > 0 \text{ such that } |\sin \frac{1}{x} - L| < \varepsilon \text{ whenever } 0 < |x| < \delta$$

- U be the statement " $\forall \varepsilon > 0 T(\varepsilon)$ " or

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that } |\sin \frac{1}{x} - L| < \varepsilon \text{ whenever } 0 < |x| < \delta$$

Then

- Fix any $\varepsilon > 0$ and any $\delta > 0$. The statement $S(\delta, \varepsilon)$ is true if all values of $\sin \frac{1}{x}$, with $0 < |x| < \delta$, lie in the interval $(L - \varepsilon, L + \varepsilon)$. As x runs over the interval $(0, \delta)$, (so that, in particular, $0 < |x| < \delta$) $\frac{1}{x}$ covers the set $(\frac{1}{\delta}, \infty)$. This contains many intervals of length 2π and hence many periods of \sin . So, as x runs over the interval $(0, \delta)$, $\sin \frac{1}{x}$ covers all of $[-1, 1]$. So $S(\delta, \varepsilon)$ is true if and only if the interval $[-1, 1]$ is contained in the interval $(L - \varepsilon, L + \varepsilon)$. In particular, when $\varepsilon < 1$, the interval $(L - \varepsilon, L + \varepsilon)$, which has length 2ε , is shorter than $[-1, 1]$ and cannot contain it, so that $S(\delta, \varepsilon)$ is false.
- Because $S(\delta, \varepsilon)$ is false for all $\delta > 0$ when $\varepsilon < 1$, $T(\varepsilon)$ is false for all $\varepsilon < 1$.
- U is false since, as we have just seen, $T(\varepsilon)$ is false for at least one $\varepsilon > 0$. For example $T(\frac{1}{2})$ is false.

In conclusion, $\sin \frac{1}{x}$ has no limit as $x \rightarrow 0$.

Theorem 5 Let $n \in \mathbb{N}$, $\vec{a}, \vec{b} \in \mathbb{R}^n$, $F, G \in \mathbb{R}$ and

$$f, g : \mathbb{R}^n \setminus \{\vec{a}\} \rightarrow \mathbb{R} \quad \vec{X} : \mathbb{R}^n \setminus \{\vec{b}\} \rightarrow \mathbb{R}^n \setminus \{\vec{a}\} \quad \gamma : \mathbb{R} \rightarrow \mathbb{R}$$

Assume that

$$\lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x}) = F \quad \lim_{\vec{x} \rightarrow \vec{a}} g(\vec{x}) = G \quad \lim_{\vec{y} \rightarrow \vec{b}} \vec{X}(\vec{y}) = \vec{a} \quad \lim_{t \rightarrow F} \gamma(t) = \gamma(F) = \Gamma$$

Then

$$(a) \quad \lim_{\vec{x} \rightarrow \vec{a}} [f(\vec{x}) + g(\vec{x})] = F + G$$

$$(b) \quad \lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x})g(\vec{x}) = FG$$

$$(c) \quad \lim_{\vec{x} \rightarrow \vec{a}} \frac{f(\vec{x})}{g(\vec{x})} = \frac{F}{G} \quad \text{if } G \neq 0$$

$$(d) \quad \lim_{\vec{y} \rightarrow \vec{b}} f(\vec{X}(\vec{y})) = F$$

$$(e) \quad \lim_{\vec{x} \rightarrow \vec{a}} \gamma(f(\vec{x})) = \Gamma$$

Proof: Note that the ε and δ in “ $\forall \varepsilon > 0 \exists \delta > 0$ such that $S(\delta, \varepsilon)$ ” are dummy variables, just as x is a dummy variable in $\int_0^1 x dx$. You may replace ε and δ by whatever symbols you like. The hypotheses of this theorem say that

$$\forall \varepsilon_f > 0 \exists \delta_f > 0 \text{ such that } |f(\vec{x}) - F| < \varepsilon_f \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta_f \quad (1)$$

$$\forall \varepsilon_g > 0 \exists \delta_g > 0 \text{ such that } |g(\vec{x}) - G| < \varepsilon_g \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta_g \quad (2)$$

$$\forall \varepsilon_X > 0 \exists \delta_X > 0 \text{ such that } |\vec{X}(\vec{y}) - \vec{a}| < \varepsilon_X \text{ whenever } 0 < |\vec{y} - \vec{b}| < \delta_X \quad (3)$$

$$\forall \varepsilon_\gamma > 0 \exists \delta_\gamma > 0 \text{ such that } |\gamma(t) - \Gamma| < \varepsilon_\gamma \text{ whenever } 0 < |t - F| < \delta_\gamma \quad (4)$$

(a) We are to prove that

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that } |f(\vec{x}) + g(\vec{x}) - F - G| < \varepsilon \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta$$

So pick any $\varepsilon > 0$. We must prove that there is a $\delta > 0$ such that

$$|f(\vec{x}) + g(\vec{x}) - F - G| < \varepsilon \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta$$

Observe that

$$|f(\vec{x}) + g(\vec{x}) - F - G| = |[f(\vec{x}) - F] + [g(\vec{x}) - G]| \leq |f(\vec{x}) - F| + |g(\vec{x}) - G|$$

Set $\varepsilon_1 = \frac{\varepsilon}{2}$ and $\varepsilon_2 = \frac{\varepsilon}{2}$. By (1) with $\varepsilon_f = \varepsilon_1$ and (2) with $\varepsilon_g = \varepsilon_2$,

$$\exists \delta_1 > 0 \text{ such that } |f(\vec{x}) - F| < \varepsilon_1 \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta_1$$

$$\exists \delta_2 > 0 \text{ such that } |g(\vec{x}) - G| < \varepsilon_2 \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta_2$$

Choose $\delta = \min\{\delta_1, \delta_2\}$. Then whenever $0 < |\vec{x} - \vec{a}| < \delta$ we also have $0 < |\vec{x} - \vec{a}| < \delta_1$ and $0 < |\vec{x} - \vec{a}| < \delta_2$ so that

$$|f(\vec{x}) + g(\vec{x}) - F - G| \leq |f(\vec{x}) - F| + |g(\vec{x}) - G| < \varepsilon_1 + \varepsilon_2 = \varepsilon$$

(b) is a homework assignment.

(c) We are to prove that

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that } \left| \frac{f(\vec{x})}{g(\vec{x})} - \frac{F}{G} \right| < \varepsilon \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta$$

So pick any $\varepsilon > 0$. We must prove that there is a $\delta > 0$ such that

$$\left| \frac{f(\vec{x})}{g(\vec{x})} - \frac{F}{G} \right| < \varepsilon \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta$$

Set $\varepsilon_1 = \frac{1}{6}|G|\varepsilon$ and $\varepsilon_2 = \frac{G^2}{6(|F|+1)}\varepsilon$. By (1) with $\varepsilon_f = \varepsilon_1$, (2) with $\varepsilon_g = \varepsilon_2$ and (2) with $\varepsilon_g = \frac{1}{2}|G|$,

$$\exists \delta_1 > 0 \text{ such that } |f(\vec{x}) - F| < \varepsilon_1 \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta_1$$

$$\exists \delta_2 > 0 \text{ such that } |g(\vec{x}) - G| < \varepsilon_2 \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta_2$$

$$\exists \delta_3 > 0 \text{ such that } |g(\vec{x}) - G| < \frac{1}{2}|G| \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta_3$$

Choose $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then whenever $0 < |\vec{x} - \vec{a}| < \delta$ we also have $0 < |\vec{x} - \vec{a}| < \delta_1$ and $0 < |\vec{x} - \vec{a}| < \delta_2$ and $0 < |\vec{x} - \vec{a}| < \delta_3$ so that

$$\begin{aligned} \left| \frac{f(\vec{x})}{g(\vec{x})} - \frac{F}{G} \right| &= \frac{|f(\vec{x})G - Fg(\vec{x})|}{|g(\vec{x})G|} = \frac{|(f(\vec{x}) - F)G - F(g(\vec{x}) - G)|}{|g(\vec{x})G|} \\ &\leq \frac{|f(\vec{x}) - F||G| + |F||g(\vec{x}) - G|}{|g(\vec{x})||G|} \\ &\leq \frac{\varepsilon_1|G| + |F|\varepsilon_2}{\frac{1}{2}|G||G|} \quad \text{since } |g(\vec{x})| = |g(\vec{x}) - G + G| \geq |G| - |g(\vec{x}) - G| \geq \frac{1}{2}|G| \\ &= \frac{1}{6}|G|\varepsilon \frac{|G|}{G^2/2} + \frac{|F|}{G^2/2} \frac{G^2}{6(|F|+1)}\varepsilon = \frac{\varepsilon}{3} + \frac{1}{3} \frac{|F|}{|F|+1} \varepsilon \\ &< \varepsilon \end{aligned}$$

(d) We are to prove that

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that } |f(\vec{X}(\vec{y})) - F| < \varepsilon \text{ whenever } 0 < |\vec{y} - \vec{b}| < \delta$$

So pick any $\varepsilon > 0$. We must prove that there is a $\delta > 0$ such that

$$|f(\vec{X}(\vec{y})) - F| < \varepsilon \text{ whenever } 0 < |\vec{y} - \vec{b}| < \delta$$

By (1) with $\varepsilon_f = \varepsilon$

$$\exists \delta_f > 0 \text{ such that } |f(\vec{x}) - F| < \varepsilon \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta_f$$

and (3) with $\varepsilon_X = \delta_f$,

$$\exists \delta_X > 0 \text{ such that } |\vec{X}(\vec{y}) - \vec{a}| < \delta_f \text{ whenever } 0 < |\vec{y} - \vec{b}| < \delta_X$$

Choosing $\delta = \delta_X$, we have

$$0 < |\vec{y} - \vec{b}| < \delta = \delta_X \implies 0 < |\vec{X}(\vec{y}) - \vec{a}| < \delta_f \implies |f(\vec{X}(\vec{y})) - F| < \varepsilon$$

(e) has essentially the same proof as part (d). We are to prove that

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ such that } |\gamma(f(\vec{x})) - \Gamma| < \varepsilon \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta$$

So pick any $\varepsilon > 0$. We must prove that there is a $\delta > 0$ such that

$$|\gamma(f(\vec{x})) - \Gamma| < \varepsilon \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta$$

By (4) with $\varepsilon_\gamma = \varepsilon$ and the hypothesis that $\gamma(F) = \Gamma$

$$\exists \delta_\gamma > 0 \text{ such that } |\gamma(t) - \Gamma| < \varepsilon \text{ whenever } |t - F| < \delta_\gamma$$

By (1) with $\varepsilon_f = \delta_\gamma$

$$\exists \delta_f > 0 \text{ such that } |f(\vec{x}) - F| < \delta_\gamma \text{ whenever } 0 < |\vec{x} - \vec{a}| < \delta_f$$

Choosing $\delta = \delta_f$, we have

$$0 < |\vec{x} - \vec{a}| < \delta = \delta_f \implies |f(\vec{x}) - F| < \delta_\gamma \implies |\gamma(f(\vec{x})) - \Gamma| < \varepsilon$$

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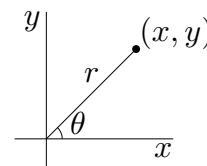
Example 6 There is a typical application of Theorem 5. Here “ $\stackrel{a}{=}$ ” means that Theorem 5.a justifies that equality.

$$\begin{aligned} \lim_{(x,y) \rightarrow (2,3)} (x + \sin y) &\stackrel{a}{=} \lim_{(x,y) \rightarrow (2,3)} x + \lim_{(x,y) \rightarrow (2,3)} \sin y \\ &\stackrel{e}{=} \lim_{(x,y) \rightarrow (2,3)} x + \sin \left(\lim_{(x,y) \rightarrow (2,3)} y \right) \\ &= 2 + \sin 3 \\ \lim_{(x,y) \rightarrow (2,3)} (x^2 y^2 + 1) &\stackrel{a}{=} \lim_{(x,y) \rightarrow (2,3)} x^2 y^2 + \lim_{(x,y) \rightarrow (2,3)} 1 \\ &\stackrel{b}{=} \left(\lim_{(x,y) \rightarrow (2,3)} x \right) \left(\lim_{(x,y) \rightarrow (2,3)} x \right) \left(\lim_{(x,y) \rightarrow (2,3)} y \right) \left(\lim_{(x,y) \rightarrow (2,3)} y \right) + 1 \\ &= 2^2 3^2 + 1 \\ \lim_{(x,y) \rightarrow (2,3)} \frac{x + \sin y}{x^2 y^2 + 1} &\stackrel{c}{=} \frac{\lim_{(x,y) \rightarrow (2,3)} (x + \sin y)}{\lim_{(x,y) \rightarrow (2,3)} (x^2 y^2 + 1)} \\ &= \frac{2 + \sin 3}{37} \end{aligned}$$

Here we have used that $\sin x$ is a continuous function. In this course we shall assume that we already know that “standard single variable calculus functions” like $\sin x$, $\cos x$, e^x and so on are continuous.

Example 7 As a second example, we consider $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^2 + y^2}$. In this example, both the numerator, $x^2 y$, and the denominator $x^2 + y^2$ tend to zero as (x, y) approaches $(0, 0)$, so we have to be more careful. A good way to see the behaviour of a function $f(x, y)$ when (x, y) is close to $(0, 0)$ is to switch to the polar coordinates r, θ using

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned}$$



Recall that the definition of $\lim_{(x,y) \rightarrow (0,0)} f(x, y) = L$ is

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{such that } |f(x, y) - L| < \varepsilon \text{ whenever } 0 < |(x, y)| < \delta \quad (5)$$

The condition $0 < |(x, y)| < \delta$ says that $0 < r < \delta$ and no restriction on θ . So substituting $x = r \cos \theta$, $y = r \sin \theta$ into (5) gives

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{such that } |f(r \cos \theta, r \sin \theta) - L| < \varepsilon \text{ whenever } 0 < r < \delta, 0 \leq \theta \leq 2\pi \quad (6)$$

For our current example

$$\frac{x^2 y}{x^2 + y^2} = \frac{(r \cos \theta)^2 (r \sin \theta)}{r^2} = r \cos^2 \theta \sin \theta$$

As $|r \cos^2 \theta \sin \theta| \leq r \rightarrow 0$ when $r \rightarrow 0$, we have

$$\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 y}{x^2 + y^2} = 0$$

Example 8 As a third example, we consider $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$. Once again, the best way to see the behaviour of $\frac{x^2 - y^2}{x^2 + y^2}$ for (x, y) close to $(0, 0)$ is to switch to polar coordinates.

$$\frac{x^2 - y^2}{x^2 + y^2} = \frac{(r \cos \theta)^2 - (r \sin \theta)^2}{r^2} = \cos^2 \theta - \sin^2 \theta = \cos(2\theta)$$

No matter how small you make $\delta > 0$, as (x, y) runs over those points with $r = |(x, y)| < \delta$, $\frac{x^2 - y^2}{x^2 + y^2}$ takes all values in the interval $[-1, 1]$. So $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

