

Directional Derivatives

The Question

Suppose that you leave the point (a, b) moving with velocity $\vec{v} = \langle v_1, v_2 \rangle$. Suppose further that the temperature at (x, y) is $f(x, y)$. Then what rate of change of temperature do you feel?

The Answers

Let's set the beginning of time, $t = 0$, to the time at which you leave (a, b) . Then at time t you are at $(a + v_1t, b + v_2t)$ and feel the temperature $f(a + v_1t, b + v_2t)$. So the change in temperature between time 0 and time t is $f(a + v_1t, b + v_2t) - f(a, b)$, the average rate of change of temperature, per unit time, between time 0 and time t is $\frac{f(a+v_1t, b+v_2t) - f(a, b)}{t}$ and the instantaneous rate of change of temperature per unit time as you leave (a, b) is $\lim_{t \rightarrow 0} \frac{f(a+v_1t, b+v_2t) - f(a, b)}{t}$. We apply the approximation

$$f(a + \Delta x, b + \Delta y) - f(a, b) \approx f_x(a, b) \Delta x + f_y(a, b) \Delta y$$

with $\Delta x = v_1t$ and $\Delta y = v_2t$. In the limit as $t \rightarrow 0$, the approximation becomes exact and we have

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{f(a+v_1t, b+v_2t) - f(a, b)}{t} &= \lim_{t \rightarrow 0} \frac{f_x(a, b) v_1t + f_y(a, b) v_2t}{t} \\ &= f_x(a, b) v_1 + f_y(a, b) v_2 \\ &= \langle f_x(a, b), f_y(a, b) \rangle \cdot \langle v_1, v_2 \rangle \end{aligned}$$

The vector $\langle f_x(a, b), f_y(a, b) \rangle$ is denoted $\vec{\nabla} f(a, b)$ and is called “the **gradient** of the function f at the point (a, b) ”. It has one component for each variable of f . The j^{th} component is the partial derivative of f with respect to the j^{th} variable, evaluated at (a, b) . The expression $\langle f_x(a, b), f_y(a, b) \rangle \cdot \langle v_1, v_2 \rangle = \vec{\nabla} f(a, b) \cdot \vec{v}$ is often denoted $D_{\vec{v}} f(a, b)$. So we conclude that *the rate of change of f per unit time as we leave (a, b) moving with velocity \vec{v}* is

$$\boxed{D_{\vec{v}} f(a, b) = \vec{\nabla} f(a, b) \cdot \vec{v}}$$

We can compute the rate of change of temperature per unit distance in a similar way. The change in temperature between time 0 and time t is $f(a + v_1t, b + v_2t) - f(a, b)$. Between time 0 and time t you have travelled a distance $|\vec{v}|t$. So the instantaneous rate of change of temperature per unit distance as you leave (a, b) is

$$\lim_{t \rightarrow 0} \frac{f(a+v_1t, b+v_2t) - f(a, b)}{t|\vec{v}|}$$

This is exactly $\frac{1}{|\vec{v}|}$ times $\lim_{t \rightarrow 0} \frac{f(a+v_1t, b+v_2t) - f(a, b)}{t}$ which we computed above to be $D_{\vec{v}} f(a, b)$. So *the rate of change of f per unit distance as we leave (a, b) moving in direction \vec{v}* is

$$\boxed{\vec{\nabla} f(a, b) \cdot \frac{\vec{v}}{|\vec{v}|} = D_{\frac{\vec{v}}{|\vec{v}|}} f(a, b)}$$

This is called the **directional derivative** of the function f at the point (a, b) in the direction \vec{v} .

Implications

We have just seen that the instantaneous rate of change of f per unit distance as we leave (a, b) moving in direction \vec{v} is

$$\vec{\nabla}f(a, b) \cdot \frac{\vec{v}}{|\vec{v}|} = |\vec{\nabla}f(a, b)| \cos \theta$$

where θ is the angle between the gradient vector $\vec{\nabla}f(a, b)$ and the direction vector \vec{v} . Since $\cos \theta$ is always between -1 and $+1$

- the direction of maximum rate of increase is that having $\theta = 0$. So to get maximum rate of increase per unit distance, as you leave (a, b) , you should move in the same direction as the gradient $\vec{\nabla}f(a, b)$. Then the rate of increase per unit distance is $|\vec{\nabla}f(a, b)|$.
- The direction of minimum (i.e. most negative) rate of increase is that having $\theta = 180^\circ$. To get minimum rate of increase per unit distance you should move in the direction opposite $\vec{\nabla}f(a, b)$. Then the rate of increase per unit distance is $-|\vec{\nabla}f(a, b)|$.
- The directions giving zero rate of increase are those perpendicular to $\vec{\nabla}f(a, b)$. If you move in a direction perpendicular to $\vec{\nabla}f(a, b)$, $f(x, y)$ remains constant as you leave (a, b) . That is, at that instant you are moving along the level curve $f(x, y) = f(a, b)$. So $\vec{\nabla}f(a, b)$ is perpendicular to the level curve $f(x, y) = f(a, b)$ at (a, b) . The corresponding statement in three dimensions is that $\vec{\nabla}F(a, b, c)$ is perpendicular to the level surface $F(x, y, z) = F(a, b, c)$ at (a, b, c) . A good way to find a vector normal to the surface $F(x, y, z) = 0$ at the point (a, b, c) is to compute the gradient $\vec{\nabla}F(a, b, c)$.

An example

Let

$$f(x, y) = 5 - x^2 - 2y^2 \quad (x_0, y_0) = (-1, -1)$$

Note that for any fixed $f_0 < 5$, $f(x, y) = f_0$ is the ellipse $x^2 + 2y^2 = 5 - f_0$. This ellipse has x -semi-axis $\sqrt{5 - f_0}$ and y -semi-axis $\sqrt{\frac{5 - f_0}{2}}$. The graph $z = f(x, y)$ consists of a bunch of horizontal ellipses stacked one on top of each other, starting with a point on the z axis when $f_0 = 5$ and increasing in size as f_0 decreases, as illustrated in the first figure below. Several level curves are sketched in the second figure below. The gradient vector

$$\nabla f(x_0, y_0) = \langle -2x, -4y \rangle \Big|_{(-1, -1)} = \langle 2, 4 \rangle = 2 \langle 1, 2 \rangle$$

at (x_0, y_0) is also illustrated in the second sketch. We have that, at (x_0, y_0)

- the direction of maximum rate of increase is $\frac{1}{\sqrt{5}} \langle 1, 2 \rangle$ and the maximum rate of increase is $|\langle 2, 4 \rangle| = 2\sqrt{5}$.
- the direction of minimum rate of increase is $-\frac{1}{\sqrt{5}} \langle 1, 2 \rangle$ and that minimum rate is $-|\langle 2, 4 \rangle| = -2\sqrt{5}$.
- the directions giving zero rate of increase are perpendicular to $\nabla f(x_0, y_0)$, that is $\pm \frac{1}{\sqrt{5}} \langle 2, -1 \rangle$. These are the directions of the tangent vector at (x_0, y_0) to the level curve of f through (x_0, y_0) .

