

The Binomial Theorem

In these notes we prove the binomial theorem, which says that for any integer $n \geq 1$,

$$(x + y)^n = \sum_{\ell=0}^n \binom{n}{\ell} x^\ell y^{n-\ell} = \sum_{\substack{\ell, m \geq 0 \\ \ell + m = n}} \binom{\ell+m}{\ell} x^\ell y^m \quad \text{where } \binom{n}{\ell} = \frac{n!}{\ell!(n-\ell)!} \quad (\text{B}_n)$$

The proof is by induction. First we check that, when $n = 1$,

$$\begin{aligned} \sum_{\ell=0}^1 \frac{1!}{\ell!(1-\ell)!} x^\ell y^{1-\ell} &= \frac{1!}{\ell!(1-\ell)!} x^\ell y^{1-\ell} \Big|_{\ell=0}^{n=1} + \frac{1!}{\ell!(1-\ell)!} x^\ell y^{1-\ell} \Big|_{\ell=1}^{n=1} = \frac{1!}{0!1!} x^0 y^1 + \frac{1!}{1!0!} x^1 y^0 \\ &= x + y \end{aligned}$$

so that (B_n) is correct for $n = 1$. To complete the proof we have to show that, for any integer $n \geq 2$, (B_n) is a consequence of (B_{n-1}) . So pick any integer $n \geq 2$ and assume that

$$(x + y)^{n-1} = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} x^\ell y^{n-1-\ell}$$

Now compute

$$(x + y)^n = (x + y)^{n-1}(x + y) = \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} x^{\ell+1} y^{n-1-\ell} + \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} x^\ell y^{n-\ell}$$

The second sum has the same powers of x and y , namely $x^\ell y^{n-\ell}$, as appear in (B_n) . To make the powers of x and y in the first sum, namely $x^{\ell+1} y^{n-1-\ell}$ look more like those of (B_n) , we make the change of summation variable from ℓ to $\tilde{\ell} = \ell + 1$. The first sum

$$\sum_{\ell=0}^{n-1} \binom{n-1}{\ell} x^{\ell+1} y^{n-1-\ell} = \sum_{\tilde{\ell}=1}^n \binom{n-1}{\tilde{\ell}-1} x^{\tilde{\ell}} y^{n-\tilde{\ell}}$$

As $\tilde{\ell}$ is just a dummy summation variable, we may call it anything we like. In particular, we may rename $\tilde{\ell}$ back to ℓ . So we now have

$$\begin{aligned} (x + y)^n &= \sum_{\ell=1}^n \binom{n-1}{\ell-1} x^\ell y^{n-\ell} + \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} x^\ell y^{n-\ell} \\ &= \binom{n-1}{\ell-1} x^\ell y^{n-\ell} \Big|_{\ell=n} + \binom{n-1}{\ell} x^\ell y^{n-\ell} \Big|_{\ell=0} + \sum_{\ell=1}^{n-1} [\binom{n-1}{\ell-1} + \binom{n-1}{\ell}] x^\ell y^{n-\ell} \end{aligned}$$

Recalling that $n! = n(n-1)!$ we have

$$\begin{aligned}\binom{n}{\ell} &= \frac{n!}{\ell!(n-\ell)!} = \frac{n(n-1)!}{\ell(\ell-1)!(n-\ell)!} = \frac{n}{\ell} \binom{n-1}{\ell-1} \\ \binom{n}{\ell} &= \frac{n!}{\ell!(n-\ell)!} = \frac{n(n-1)!}{\ell!(n-\ell)(n-\ell-1)!} = \frac{n}{n-\ell} \binom{n-1}{\ell}\end{aligned}$$

So

$$\begin{aligned}(x+y)^n &= \binom{n-1}{n-1}x^n + \binom{n-1}{0}y^n + \sum_{\ell=1}^{n-1} [\binom{n-1}{\ell-1} + \binom{n-1}{\ell}]x^\ell y^{n-\ell} \\ &= x^n + y^n + \sum_{\ell=1}^{n-1} \binom{n}{\ell} \left[\frac{\ell}{n} + \frac{n-\ell}{n} \right] x^\ell y^{n-\ell} \\ &= x^n + y^n + \sum_{\ell=1}^{n-1} \binom{n}{\ell} x^\ell y^{n-\ell} \\ &= \sum_{\ell=0}^n \binom{n}{\ell} x^\ell y^{n-\ell}\end{aligned}$$

as desired.