In this example, we find the mass of the part of the first octant that is inside the sphere $x^2 + y^2 + z^2 = a^2$, if the density is $\rho(x, y, z) = xy$.

- Slice the solid into horizontal plates by inserting many planes of constant $z$, with the various values of $z$ differing by $dz$.

  - Each plate has thickness $dz$, and
  - has $z$ almost constant throughout the plate (it only varies by $dz$), and
  - has $(x, y)$ running over $x \geq 0, y \geq 0, x^2 + y^2 \leq a^2 - z^2$.
  - The bottom plate starts at $z = 0$ and the top plate ends at $z = a$.

- Concentrate on any one plate. Subdivide it into long thin “square” beams by inserting many planes of constant $y$, with the various values of $y$ differing by $dy$.

  - Each beam has cross-sectional area $dy\,dz$, and
  - has $y$ and $z$ essentially constant throughout the beam, and
  - has $x$ running over $0 \leq x \leq \sqrt{a^2 - y^2 - z^2}$.
  - The leftmost beam has $y = 0$ and the rightmost beam has $y = \sqrt{a^2 - z^2}$.
Concentrate on any one beam. Subdivide it into tiny approximate “cubes” by inserting many planes of constant $x$, with the various values of $x$ differing by $dx$.

- Each cube has volume $dx \, dy \, dz$, and
- has $x$, $y$ and $z$ all essentially constant throughout the cube.
- The first cube has $x = 0$ and the last cube has $x = \sqrt{a^2 - y^2 - z^2}$.

• Now we can build up the mass.
  - Concentrate on one approximate cube, say the cube containing the point $(x, y, z)$.
    - That cube has volume essentially $dV = dx \, dy \, dz$ and
    - essentially has density $\rho(x, y, z) = xy$ and so
    - essentially has mass $\rho(x, y, z) \, dV = xy \, dx \, dy \, dz$.
  - To get the mass of any one beam, say the beam in the figure above, we just add up the masses of the approximate cubes in that beam, by integrating $x$ from its smallest value on the beam, namely 0, to its largest value on the beam, namely $\sqrt{a^2 - y^2 - z^2}$. The mass of the beam is thus
    \[ dy \, dz \int_0^{\sqrt{a^2 - y^2 - z^2}} dx \, xy \]
  - To get the mass of any one plate, say the plate in the figure above, we just add up the masses of the beams in that plate, by integrating $y$ from its smallest value on the plate, namely 0, to its largest value on the plate, namely $\sqrt{a^2 - z^2}$. The mass of the plate is thus
    \[ dz \int_0^{\sqrt{a^2 - z^2}} dy \int_0^{\sqrt{a^2 - y^2 - z^2}} dx \, xy \]
  - To get the mass of the whole solid, we just add up the masses of the plates that it contains, by integrating $z$ from its smallest value, namely 0, to its largest value on the solid, namely $a$.\[ \int_0^a dz \int_0^{\sqrt{a^2 - z^2}} dy \int_0^{\sqrt{a^2 - y^2 - z^2}} dx \, xy \]
The mass of the solid is thus

\[ \int_0^a dz \int_0^{\sqrt{a^2-z^2}} dy \int_0^{\sqrt{a^2-y^2-z^2}} dx \ xy = \int_0^a dz \int_0^{\sqrt{a^2-z^2}} dy \ y \left[ \int_0^{\sqrt{a^2-y^2-z^2}} x \ dx \right] \]

\[ = \int_0^a dz \int_0^{\sqrt{a^2-z^2}} dy \ y \left( \frac{y}{2} \right) \left( a^2 - y^2 - z^2 \right) \]

\[ = \int_0^a dz \left[ \int_0^{\sqrt{a^2-z^2}} \left( \frac{(a^2 - z^2) y}{2} - \frac{y^2}{2} \right) \ dy \right] \]

\[ = \int_0^a \left[ \frac{(a^2 - z^2)}{2} \int_0^{\sqrt{a^2-z^2}} \left( \frac{(a^2 - z^2)^2}{4} - \frac{(a^2 - z^2)^2}{8} \right) \right] \ dz \]

\[ = \frac{1}{8} \int_0^a \left[ a^4 - 2a^2 z^2 + z^4 \right] \ dz \]

\[ = \frac{1}{8} \left[ a^5 - \frac{2}{3} a^5 + \frac{1}{5} a^5 \right] \]

\[ = \frac{a^5}{8} \left( 15 - 10 + 3 \right) \]

\[ = \frac{a^5}{15} \]