Global Maximum/Minimum Example

Problem. Find the maximum and minimum values of
\[ f(x, y) = xy + 2x + y \]
on the triangular region with vertices (0, 0), (1, 0) and (0, 2).

Solution.

Interior. Since
\[ f_x(x, y) = y + 2 \quad f_y(x, y) = x + 1 \]
there are no singular points and the only critical point, \((-1, -2)\), is not in the triangular region.

Side. \(x = 0, \ 0 \leq y \leq 2\). On that side \(f(0, y) = y\) and
\[ \min_{0 \leq y \leq 2} f(0, y) = f(0, 0) = 0 \quad \max_{0 \leq y \leq 2} f(0, y) = f(0, 2) = 2 \]

Base. \(y = 0, \ 0 \leq x \leq 1\). On that side \(f(x, 0) = 2x\) and
\[ \min_{0 \leq x \leq 1} f(x, 0) = f(0, 0) = 0 \quad \max_{0 \leq x \leq 1} f(x, 0) = f(1, 0) = 2 \]

Hypotenuse. On that side \(y = 2 - 2x, \ 0 \leq x \leq 1\) and
\[ f(x, 2 - 2x) = x(2 - 2x) + 2x + (2 - 2x) = -2x^2 + 2x + 2 \]
Write \(g(x) = -2x^2 + 2x + 2\). The maximum and minimum of \(g(x)\) for \(0 \leq x \leq 1\), and hence the maximum and minimum values of \(f\) on the hypotenuse of the triangle, must be achieved either at
- \(x = 0, \) where \(f(0, 2) = g(0) = 2, \) or at
- \(x = 1, \) where \(f(1, 0) = g(1) = 2, \) or when
- \(0 = g'(x) = -4x + 2\) so that \(x = \frac{1}{2}, y = 2 - 2(\frac{1}{2}) = 1\) and
\[ f\left(\frac{1}{2}, 1\right) = g\left(\frac{1}{2}\right) = -2\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) + 2 = \frac{5}{2} \]

Candidates. Here are all the candidates for the location of a max or min.

<table>
<thead>
<tr>
<th>point</th>
<th>(0, 0)</th>
<th>(0, 2)</th>
<th>(1, 0)</th>
<th>(\frac{1}{2}, 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>value of (f)</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>(\frac{5}{2})</td>
</tr>
<tr>
<td>min</td>
<td></td>
<td></td>
<td></td>
<td>max</td>
</tr>
</tbody>
</table>
Finding the Equation of the Line Through (0, 2) and (1, 0)

Method 1: using $Ax + By = 1$.
- Every line in the $xy$-plane has an equation of the form $ax + by = c$.
- In this case (0, 0) is not on the line so that $c \neq 0$ and we can divide the equation by $c$, giving $\frac{a}{c}x + \frac{b}{c}y = 1$. Rename $\frac{a}{c} = A$ and $\frac{b}{c} = B$.
- (0, 2) is on the line so that $Ax \big|_{x=0} + By \big|_{y=2} = 1 \implies B = \frac{1}{2}$
- (1, 0) is on the line so that $Ax \big|_{x=1} + By \big|_{y=0} = 1 \implies A = 1$
- So the line is $x + \frac{y}{2} = 1$ or $y = 2 - 2x$.

Method 2: using $y = mx + b$.
- $b$ is the $y$-intercept, i.e. the $y$-coordinate of the point on the line where $x = 0$. In this case $b = 2$.
- $m$ is the slope. In this case $m = \frac{\Delta y}{\Delta x} = \frac{0 - 2}{1 - 0} = -2$.
- So the line is $y = 2 - 2x$.

Method 2: using parameterization.
- (0, 2) is one point on the line.
- the vector from (0, 2) to (1, 0), namely $\langle 1 - 0, 0 - 2 \rangle = \langle 1, -2 \rangle$, is a direction vector for the line.
- So the line is $\langle x - 0, y - 2 \rangle = t \langle 1, -2 \rangle$ or $x = t$, $y = 2 - 2t$. 