Maximum/Minimum Examples

Problem. Find and classify all the critical points of $f(x, y) = x^3 + xy^2 - 3x^2 - 4y^2 + 4$.

Solution.

$$
\begin{aligned}
f &= x^3 + xy^2 - 3x^2 - 4y^2 + 4 \\
f_x &= 3x^2 + y^2 - 6x \qquad f_{xx} = 6x - 6 \qquad f_{xy} = 2y \\
f_y &= 2xy - 8y \qquad\qquad f_{yy} = 2x - 8
\end{aligned}
$$

The critical points are the solutions of

$$
\begin{aligned}
& f_x = 0 && \text{and} \quad f_y = 0 \\
\iff \quad & 3x^2 + y^2 - 6x = 0 && \text{and} \quad 2y(x-4) = 0 \\
\iff \quad & 3x^2 + y^2 - 6x = 0 && \text{and} \quad \{\ y = 0 \text{ or } x = 4\ \}
\end{aligned}
$$

When $y = 0$, x must obey $0 = 3x^2 - 6x = 3x(x-2)$ so that $x = 0$ or $x = 2$.
When $x = 4$, y must obey $0 = 3 \times 4^2 + y^2 - 6 \times 4 = 24 + y^2$, which is impossible,
So, there are two critical points: $(0, 0)$, $(2, 0)$.

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0, 0)$	$(-6) \times (-8) - 0^2 > 0$	$-6 < 0$	local max
$(2, 0)$	$6 \times (-4) - 0^2 < 0$		saddle pt

Problem. Find the maximum and minimum values of $f(x,y) = x^3 + xy^2 - 3x^2 - 4y^2 + 4$ on $x^2 + y^2 \le 1$.

Solution.

Case 1: $x^2 + y^2 < 1$. If f takes its maximum or minimum value at a point in the interior, $x^2 + y^2 < 1$, then that point must be a critical point of f. In the last example, we found that only critical points of f are $(0,0)$, which lies in $x^2+y^2 < 1$, and $(2,0)$, which does not lie in $x^2 + y^2 < 1$.

Case 2: $x^2 + y^2 = 1$. When $x^2 + y^2 = 1$, $y^2 = 1 - x^2$ and

$$f = x^3 + x(1-x^2) - 3x^2 - 4(1-x^2) + 4 = x + x^2$$

The max and min of $x + x^2$ for $-1 \le x \le 1$ must occur either when $x = -1$ ($\Rightarrow y = f = 0$) or when $x = +1$ ($\Rightarrow y = 0$, $f = 2$) or when $0 = \frac{d}{dx}(x + x^2) = 1 + 2x$ ($\Rightarrow x = -\frac{1}{2}$, $y = \pm\sqrt{\frac{3}{4}}$, $f = -\frac{1}{4}$).

Here are all the candidates for the location of a max or min.

point	$(0,0)$	$(-1,0)$	$(1,0)$	$\left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right)$
value of f	4	2	0	$-\frac{1}{4}$
	max			min