Math 200 Test 3 Solutions **October 25, 2019**

1. Suppose that $f(x,y)$, $v(s,t)$ and $w(s,t)$ obey $v(1,3)=2$, $w(1,3)=4$ and

$$\frac{\partial f}{\partial x}(1,3)=-5 \quad \frac{\partial f}{\partial x}(2,4)=9 \quad \frac{\partial v}{\partial s}(1,3)=4 \quad \frac{\partial v}{\partial s}(2,4)=5 \quad \frac{\partial w}{\partial s}(1,3)=-1 \quad \frac{\partial w}{\partial s}(2,4)=1$$
$$\frac{\partial f}{\partial y}(1,3)=-6 \quad \frac{\partial f}{\partial y}(2,4)=10 \quad \frac{\partial v}{\partial t}(1,3)=3 \quad \frac{\partial v}{\partial t}(2,4)=6 \quad \frac{\partial w}{\partial t}(1,3)=2 \quad \frac{\partial w}{\partial t}(2,4)=7$$

Let $g(s,t)=f\big(v(s,t),w(s,t)\big)$. Find $\frac{\partial g}{\partial s}(1,3)$.

Solution. By the chain rule

$$\begin{aligned}\frac{\partial g}{\partial s}(s,t) &= \frac{\partial}{\partial s} f\big(v(s,t),w(s,t)\big)\\ &= \frac{\partial f}{\partial x}\big(v(s,t),w(s,t)\big)\frac{\partial v}{\partial s}(s,t)+\frac{\partial f}{\partial y}\big(v(s,t),w(s,t)\big)\frac{\partial w}{\partial s}(s,t)\end{aligned}$$

In particular

$$\begin{aligned}\frac{\partial g}{\partial s}(1,3) &= \frac{\partial f}{\partial x}\overbrace{\big(v(1,3),w(1,3)\big)}^{(2,4)}\frac{\partial v}{\partial s}(1,3)+\frac{\partial f}{\partial y}\overbrace{\big(v(1,3),w(1,3)\big)}^{(2,4)}\frac{\partial w}{\partial s}(1,3)\\ &= 9\times 4+10\times(-1)=\boxed{26}\end{aligned}$$

2. The height of land in the vicinity of a hill is given, in terms of horizontal coordinates x and y, by

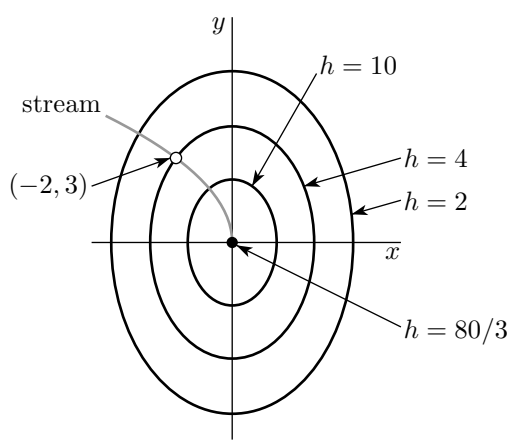
$$h(x,y)=\frac{80}{3+2x^2+y^2}$$

A hiker is standing by a small stream at $x=-2,\ y=3$.
(a) In what direction $\hat{\mathbf{s}}=\langle s_x,s_y\rangle$ is the stream flowing near the hiker? Make $\hat{\mathbf{s}}$ a unit vector.
(b) How rapidly is the stream descending there. That is, how many meters does it drop per horizontal meter travelled?
(c) Draw a contour map showing the height of land in the vicinity of the hill. Show at least three level curves and show the stream on your map.
(d) At what angle to the path of the stream (on the map) should the hiker set out in order to descend the hill at a 45° slope (that is, to drop one meter of altitude per horizontal meter travelled)?

Solution. (a) The gradient of h is $\boldsymbol{\nabla} h(x,y)=-\frac{80}{(3+2x^2+y^2)^2}\langle 4x,2y\rangle$. In particular, for $x=-2,\ y=3$, $\boldsymbol{\nabla} h(-2,3)=-\frac{80}{(3+8+9)^2}\langle -8,6\rangle=-\frac{1}{5}\langle -8,6\rangle=\frac{2}{5}\langle 4,-3\rangle$. The stream flows in the direction of maximum rate of descent, which is directly opposite $\boldsymbol{\nabla} h(-2,3)$. A unit vector in that direction is $\boxed{\hat{\mathbf{s}}=\frac{1}{5}\langle -4,3\rangle}$.

(b) The rate of descent is $|\boldsymbol{\nabla} h(-2,3)|=\frac{2}{5}\sqrt{16+9}=\boxed{2}$ meters of drop per meter of horizontal travel.

(c) The level curves have equations of the form $h(x,y)=\frac{80}{3+2x^2+y^2}=C$ or $2x^2+y^2=\frac{80}{C}-3$. This is an ellipse with x semi-axis $\frac{1}{\sqrt{2}}\sqrt{\frac{80}{C}-3}$ and y semi-axis $\sqrt{\frac{80}{C}-3}$ (which is larger than the x semi-axis). Here is a sketch. Note that the stream crosses the level curves at right angles.



(d) In order to have a rate of descent of one vertical meter per meter of horizontal travel we need to move in a direction $\hat{\mathbf{d}}$ with $|\hat{\mathbf{d}}| = 1$ and $-1 = D_{\hat{\mathbf{d}}}h(-2,3) = \boldsymbol{\nabla} h(-2,3) \cdot \hat{\mathbf{d}} = \frac{2}{5}\langle 4, -3\rangle \cdot \hat{\mathbf{d}}$ or $1 = 2\hat{\mathbf{s}} \cdot \hat{\mathbf{d}}$. Denoting by θ the angle between $\hat{\mathbf{s}}$ and $\hat{\mathbf{d}}$, we need $1 = 2|\hat{\mathbf{s}}||\hat{\mathbf{d}}|\cos\theta$ or $\cos\theta = \frac{1}{2}$ or $\boxed{\theta = 60^\circ}$.

3. Find the point or points on the surface $x^2 + y^2 + z^2 + xy + xz = 6$ which are farthest from the xy-plane.

Solution 1. Let (x_0, y_0, z_0) be a point of the given surface which is farthest from the the xy–plane (i.e. has maximum $|z|$). Write $G(x,y,z) = x^2 + y^2 + z^2 + xy + xz$ so that the surface is $G(x,y,z) = 6$. At (x_0, y_0, z_0) the tangent plane to the surface is parallel to the xy-plane, or equivalently, the normal vector to the surface has vanishing x and y components. The normal vector is $\boldsymbol{\nabla}(x^2 + y^2 + z^2 + xy + xz)\big|_{(x_0,y_0,z_0)} = \langle 2x_0 + y_0 + z_0, 2y_0 + x_0, 2z_0 + x_0\rangle$, so

$$G_x(x_0, y_0, z_0) = 2x_0 + y_0 + z_0 = 0 \tag{1}$$
$$G_y(x_0, y_0, z_0) = 2y_0 + x_0 = 0 \tag{2}$$
$$G(x_0, y_0, z_0) = x_0^2 + y_0^2 + z_0^2 + x_0y_0 + x_0z_0 = 6 \tag{3}$$

Subbing $x_0 = -2y_0$, from (2), into (1), gives $2(-2y_0) + y_0 + z_0 = 0$ or $z_0 = 3y_0$. Then (3) gives

$$6 = (-2y_0)^2 + y_0^2 + (3y_0)^2 + (-2y_0)y_0 + (-2y_0)(3y_0) = 6y_0^2 \quad \Longrightarrow \quad y_0 = \pm 1$$

So, $(x_0, y_0, z_0) = (-2y_0, y_0, 3y_0)\big|_{y_0=\pm 1} = \boxed{\pm(-2,1,3)}$. Both these points are of distance 3 from the xy-plane, so both are farthest points.

Solution 2. Let (x_0, y_0, z_0) be a point of the given surface which is farthest from the the xy–plane. Near (x_0, y_0, z_0) the surface is $z = z(x,y)$ with the function $z(x,y)$ obeying

$$x^2 + y^2 + z(x,y)^2 + xy + x\,z(x,y) = 6 \tag{4}$$

for all (x,y) near (x_0, y_0). Taking the partial derivative of equation (4) with respect to x gives

$$2x + 2z(x,y)\,z_x(x,y) + y + z(x,y) + x\,z_x(x,y) = 0 \tag{5}$$

Taking the partial derivative of equation (4) with respect to y gives

$$2y + 2z(x,y)\,z_y(x,y) + x + x\,z_y(x,y) = 0 \tag{6}$$

As $z(x,y)$ is maximized (or minimized) at (x_0, y_0, z_0) we have $z_x(x_0, y_0) = 0$ and $z_y(x_0, y_0) = 0$ and $z(x_0, y_0) = z_0$ so that, when $x = x_0$ and $y = y_0$, equations (5) and (6) simplify to

$$2x_0 + y_0 + z_0 = 0 \tag{1}$$
$$2y_0 + x_0 = 0 \tag{2}$$

We can now continue as in Solution 1.