**Math 200 Test 2 Solutions** **October 11, 2019**

1. Denote by $S$ the level surface of
$$f(x,y,z) = z\sin(x+y) + y\cos z$$
that passes through $(\pi,\pi,\pi)$.
(a) Find the tangent plane to $S$ at $(\pi,\pi,\pi)$.
(b) Find the point of intersection between the $yz$-plane and the normal line to $S$ at $(\pi,\pi,\pi)$.

**Solution.** A normal vector to $S$ at $(\pi,\pi,\pi)$ is
$$\begin{aligned}\boldsymbol{\nabla} f(\pi,\pi,\pi) &= \langle z\cos(x+y)\ ,\ z\cos(x+y)+\cos z\ ,\ \sin(x+y) - y\sin z\rangle\big|_{(\pi,\pi,\pi)}\\ &= \langle \pi\ ,\ \pi-1\ ,\ 0\rangle\end{aligned}$$

(a) The tangent plane has the equation
$$\boxed{\pi(x-\pi)+(\pi-1)(y-\pi)=0} \text{ or } \boxed{\pi x+(\pi-1)y = -\pi+2\pi^2}$$

(b) The normal line is
$$\langle x-\pi, y-\pi, z-\pi\rangle = t\,\langle \pi\ ,\ \pi-1\ ,\ 0\rangle \qquad \text{or} \qquad (x,y,z) = \big(\pi+t\pi\ ,\ \pi+t(\pi-1)\ ,\ \pi\big)$$

The normal line intersects the $yz$-plane when
$$x=0 \iff \pi+t\pi=0 \iff t=-1 \iff \boxed{(x,y,z)=(0,1,\pi)}$$

**Warning:**
The normal vector to the surface $G(x,y,z)=0$ at $(x_0,y_0,z_0)$ is $\boldsymbol{\nabla} G(x_0,y_0,z_0)$, **not** $\boldsymbol{\nabla} G(x,y,z)$. See Warning 2.5.9 in the CLP-3 text.

2. Define the function
$$f(x,y) = \begin{cases} \dfrac{x^2+2x+y^2-y-1}{x+y-1} & \text{if } x+y\neq 1\\ 3 & \text{if } x+y=1\end{cases}$$
(a) Evaluate, if possible, $\frac{\partial f}{\partial y}(1,1)$.
(b) Evaluate, if possible, $\frac{\partial f}{\partial y}(-1,2)$.

**Solution.** (a) If $x+y\neq 1$,
$$\tfrac{\partial f}{\partial y}(x,y) = \tfrac{\partial}{\partial y}\big(\tfrac{x^2+2x+y^2-y-1}{x+y-1}\big) = \tfrac{2y-1}{x+y-1} - \tfrac{x^2+2x+y^2-y-1}{(x+y-1)^2}$$
In particular,
$$\tfrac{\partial f}{\partial y}(1,1) = \tfrac{2\times1-1}{1+1-1} - \tfrac{1+2\times1+1^2-1-1}{(1+1-1)^2} = 1-2 = \boxed{-1}$$

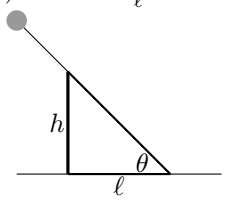
(b) By definition,
$$\begin{aligned}\tfrac{\partial f}{\partial y}(-1,2) &= \lim_{h\to0}\tfrac{f(-1,2+h)-f(-1,2)}{h} = \lim_{h\to0}\tfrac{1}{h}\Big[\tfrac{1-2+(2+h)^2-(2+h)-1}{-1+(2+h)-1} - 3\Big]\\ &= \lim_{h\to0}\tfrac{1}{h}\Big[\tfrac{h^2+3h}{h}-3\Big] = \lim_{h\to0}\tfrac{1}{h}\big[h\big] = \lim_{h\to0}1 = \boxed{1}\end{aligned}$$

**Warnings:**
(a) Knowing the value of a function at one point tells you nothing about the value of any derivative of that function at that point. In particular, knowing that $f(-1,2)=3$ *does* not necessarily mean that $f$ is a constant, so that $f_y(-1,2)=0$. For example, the function $f(x,y)=3+7(y-2)$ obeys $f(-1,2)=3$, but $f_y(-1,2)=7$.

(b) $f_y(-1,2)$ is the limit of $f_y(x,y)$ as $(x,y)$ approaches $(-1,2)$ **only** if $f_y(x,y)$ is continuous at $(-1,2)$. In this question, $f_y(x,y)$ is not continuous at $(-1,2)$ and cannot be computed as the limit of $\frac{2y-1}{x+y-1} - \frac{x^2+2x+y^2-y-1}{(x+y-1)^2}$ as $(x,y)$ approaches $(-1,2)$.

3. A tree whose height is measured to be $30 \pm 1$m casts a shadow (on flat horizontal ground) of length $30 \pm 0.5$m. From this, the angle, $\theta$, of the sun above the horizon is calculated. What is the approximate percentage error in the computed value of $\theta$?

**Solution.** The angle $\theta(h,\ell) = \arctan \frac{h}{\ell}$. Since




$$\frac{\partial\theta}{\partial h} = \frac{1/\ell}{1+(h/\ell)^2} = \frac{1}{60} \text{ when } h = \ell = 30$$
$$\frac{\partial\theta}{\partial \ell} = \frac{-h/\ell^2}{1+(h/\ell)^2} = -\frac{1}{60} \text{ when } h = \ell = 30$$

we have

$$\theta(30+\Delta h, 30+\Delta\ell) \approx \theta(30,30) + \frac{1}{60}\Delta h - \frac{1}{60}\Delta\ell$$

The (approximate) maximum error $\left|\frac{1}{60}\Delta h - \frac{1}{60}\Delta\ell\right|$ is gotten by taking $\Delta h = 1$ and $\Delta\ell = -0.5$ (or $\Delta h = -1$ and $\Delta\ell = 0.5$) and is $\frac{3}{2\times 60} = \frac{1}{40}$. When $h = \ell = 30$, $\theta = \frac{\pi}{4}$, so that the corresponding percentage error is $100\frac{1}{40}\frac{4}{\pi} = \boxed{\frac{10}{\pi}} \approx 3.2$.

**Warnings:**

(a) The standard formulae for derivatives of trig finctions, like $\frac{d}{dx}\sin x = \cos x$ and $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$, are valid only for angles expressed in radians (not degrees). That is why the percentage error is $100\frac{1/40}{\pi/4}$ rather than $100\frac{1/40}{45}$.

(b) We are only told that $-1 \le \Delta h \le 1$ and $-0.5 \le \Delta\ell \le 0.5$. We are told nothing about the signs of $\Delta h$ and $\Delta\ell$. In particular, they need not be of the same sign. Choosing $\Delta h = 1$ and $\Delta\ell = 0.5$ gives $\frac{1}{60}\Delta h - \frac{1}{60}\Delta\ell = \frac{1}{120}$ which is **not** the maximum error.