1. Denote by $S$ the level surface of 
$$f(x,y,z) = z \sin(x+y) + y \cos z$$
that passes through $(\pi,\pi,\pi)$.
(a) Find the tangent plane to $S$ at $(\pi,\pi,\pi)$.
(b) Find the point of intersection between the $yz$-plane and the normal line to $S$ at $(\pi,\pi,\pi)$.

**Solution.** A normal vector to $S$ at $(\pi,\pi,\pi)$ is 
$$\nabla f(\pi,\pi,\pi) = \langle z \cos(x+y), z \cos(x+y) + \cos z, \sin(x+y) - y \sin z \rangle \big|_{(\pi,\pi,\pi)} = \langle \pi, \pi-1, 0 \rangle \nabla f(\pi,\pi,\pi)$$

(a) The tangent plane has the equation
$$\pi(x - \pi) + (\pi - 1)(y - \pi) = 0 \quad \text{or} \quad \pi x + (\pi - 1)y = -\pi + 2\pi^2$$

(b) The normal line is
$$\langle x - \pi, y - \pi, z - \pi \rangle = t \langle \pi, \pi - 1, 0 \rangle \quad \text{or} \quad (x, y, z) = (\pi t, \pi + t(\pi - 1), \pi)$$

The normal line intersects the $yz$-plane when
$$x = 0 \iff \pi + t \pi = 0 \iff t = -1 \iff (x, y, z) = (0, 1, \pi)$$

**Warning:**
The normal vector to the surface $G(x,y,z) = 0$ at $(x_0, y_0, z_0)$ is $\nabla G(x_0, y_0, z_0)$, not $\nabla G(x,y,z)$. See Warning 2.5.9 in the CLP-3 text.

2. Define the function
$$f(x,y) = \begin{cases} 
\frac{x^2 + 2x + y^2 - y - 1}{x+y-1} & \text{if } x + y \neq 1 \\
3 & \text{if } x + y = 1 
\end{cases}$$

(a) Evaluate, if possible, $\frac{\partial f}{\partial y}(1,1)$.
(b) Evaluate, if possible, $\frac{\partial f}{\partial y}(-1,2)$.

**Solution.** (a) If $x + y \neq 1$,
$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial y} \left( \frac{x^2 + 2x + y^2 - y - 1}{x+y-1} \right) = \frac{2y-1}{x+y-1} - \frac{x^2 + 2x + y^2 - y - 1}{(x+y-1)^2}$$

In particular,
$$\frac{\partial f}{\partial y}(1,1) = \frac{2 \cdot 1 - 1}{1+1-1} - \frac{1+2+1+1^2-1}{(1+1-1)^2} = 1 - 2 = -1$$

(b) By definition,
$$\frac{\partial f}{\partial y}(-1,2) = \lim_{h \to 0} \frac{f(-1,2+h) - f(-1,2)}{h} = \lim_{h \to 0} \frac{\left[ \frac{1+2(2+h)^2-(2+h) - 1}{1+(2+h)^2-1} - 3 \right]}{h}$$
$$= \lim_{h \to 0} \frac{h^2 + 3h}{h} = \lim_{h \to 0} \frac{h^2 + 3h}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$

**Warnings:**
(a) Knowing the value of a function at one point tells you nothing about the value of any derivative of that function at that point. In particular, knowing that $f(-1,2) = 3$ does not necessarily mean that $f$ is a constant, so that $f_y(-1,2) = 0$. For example, the function $f(x,y) = 3 + 7(y-2)$ obeys $f(-1,2) = 3$, but $f_y(-1,2) = 7$. 
(b) \( f_y(-1, 2) \) is the limit of \( f_y(x, y) \) as \((x, y)\) approaches \((-1, 2)\) **only** if \( f_y(x, y) \) is continuous at \((-1, 2)\).

In this question, \( f_y(x, y) \) is not continuous at \((-1, 2)\) and cannot be computed as the limit of

\[
\frac{xy-1}{x+y-1}
\]

as \((x, y)\) approaches \((-1, 2)\).

3. A tree whose height is measured to be \(30 \pm 1\) m casts a shadow (on flat horizontal ground) of length \(30 \pm 0.5\) m.

From this, the angle, \( \theta \), of the sun above the horizon is calculated. What is the approximate percentage error in the computed value of \( \theta \)?

**Solution.** The angle \( \theta(h, \ell) = \arctan \frac{h}{\ell} \). Since

\[
\frac{\partial \theta}{\partial h} = \frac{1/\ell}{1+(h/\ell)^2} = \frac{1}{60} \quad \text{when} \; h = \ell = 30
\]

\[
\frac{\partial \theta}{\partial \ell} = \frac{-h/\ell^2}{1+(h/\ell)^2} = -\frac{1}{60} \quad \text{when} \; h = \ell = 30
\]

we have

\[
\theta(30 + \Delta h, 30 + \Delta \ell) \approx \theta(30, 30) + \frac{1}{60} \Delta h - \frac{1}{60} \Delta \ell
\]

The (approximate) maximum error \( \left| \frac{1}{60} \Delta h - \frac{1}{60} \Delta \ell \right| \) is gotten by taking \( \Delta h = 1 \) and \( \Delta \ell = -0.5 \) (or \( \Delta h = -1 \) and \( \Delta \ell = 0.5 \)) and is \( \frac{3}{2 \times 60} = \frac{1}{40} \). When \( h = \ell = 30, \; \theta = \frac{\pi}{4} \), so that the corresponding percentage error is \( 100 \frac{1/40}{\pi/4} \approx 3.2 \).

**Warnings:**

(a) The standard formulae for derivatives of trig functions, like \( \frac{d}{dx} \sin x = \cos x \) and \( \frac{d}{dx} \arctan x = \frac{1}{1+x^2} \), are valid only for angles expressed in radians (not degrees). That is why the percentage error is \( 100 \frac{1/40}{\pi/4} \) rather than \( 100 \frac{1/40}{180/\pi} \).

(b) We are only told that \(-1 \leq \Delta h \leq 1\) and \(-0.5 \leq \Delta \ell \leq 0.5\). We are told nothing about the signs of \( \Delta h \) and \( \Delta \ell \). In particular, they need not be of the same sign. Choosing \( \Delta h = 1 \) and \( \Delta \ell = 0.5 \) gives \( \frac{1}{60} \Delta h - \frac{1}{60} \Delta \ell = \frac{1}{120} \) which is **not** the maximum error.