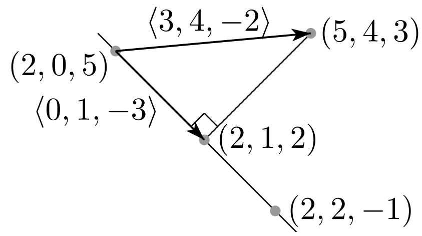
Math 200 Test 1 Solutions **September 20, 2019**

1. Drop a perpendicular from the point $(5, 4, 3)$ to the line L which passes through the points $(2, 0, 5)$ and $(2, 2, -1)$. Where does the perpendicular hit L?

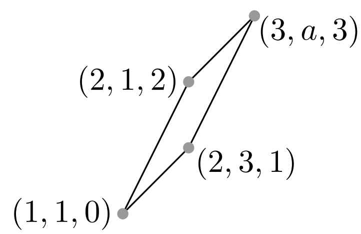
   Solution. The vector from $(2, 0, 5)$ to $(5, 4, 3)$ is $\langle 5-2, 4-0, 3-5\rangle = \langle 3, 4, -2\rangle$. The vector $\langle 2-2, 2-0, -1-5\rangle = \langle 0, 2, -6\rangle$ lies in the line. The projection of $\langle 3, 4, -2\rangle$ on $\langle 0, 2, -6\rangle$ is

$$\frac{\langle 3,4,-2\rangle\cdot\langle 0,2,-6\rangle}{|\langle 0,2,-6\rangle|^2}\langle 0, 2, -6\rangle = \frac{20}{40}\langle 0, 2, -6\rangle = \langle 0, 1, -3\rangle$$

   The perpendicular hits L at $(2+0, 0+1, 5-3) = \boxed{(2, 1, 2)}$.



2. For some number a, the points $(1, 1, 0)$, $(2, 3, 1)$, $(2, 1, 2)$, $(3, a, 3)$ are the vertices of a parallelogram.
   (a) What is a? Include your reasoning.
   (b) What is the area of the parallelogram?



   Solution. (a) For the points to be the vertices of a parallelogram, the vector from $(1, 1, 0)$ to $(2, 3, 1)$, namely $\langle 2-1, 3-1, 1-0\rangle = \langle 1, 2, 1\rangle$, must be the same as the vector from $(2, 1, 2)$ to $(3, a, 3)$, namely $\langle 3-2, a-1, 3-2\rangle = \langle 1, a-1, 1\rangle$. So $\boxed{a = 3}$.
   (b) The vector from $(1, 1, 0)$ to $(2, 3, 1)$, namely $\langle 2-1, 3-1, 1-0\rangle = \langle 1, 2, 1\rangle$, and the vector from $(1, 1, 0)$ to $(2, 1, 2)$, namely $\langle 2-1, 1-1, 2-0\rangle = \langle 1, 0, 2\rangle$ form two sides of the parallelogram and

$$\begin{aligned}\langle 1, 2, 1\rangle \times \langle 1, 0, 2\rangle &= \det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}}\\ 1 & 2 & 1\\ 1 & 0 & 2\end{bmatrix}\\ &= \hat{\boldsymbol{\imath}}(2\times 2 - 1\times 0) - \hat{\boldsymbol{\jmath}}(1\times 2 - 1\times 1) + \hat{\mathbf{k}}(1\times 0 - 2\times 1)\\ &= 4\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} - 2\hat{\mathbf{k}}\end{aligned}$$

   So the area of the parallelogram is $|4\hat{\boldsymbol{\imath}} - 1\hat{\boldsymbol{\jmath}} + 2\hat{\mathbf{k}}| = \sqrt{16+1+4} = \boxed{\sqrt{21}}$.

3. Consider the surface $x^2 - 2y + z^2 = 4$.
   (a) Describe and sketch the slice of this surface with $y = 0$.
   (b) Sketch the surface. Give the reasoning you used to arrive at your sketch.

Solution. (a) When $y = 0$, $x^2 - 2y + z^2 = x^2 + z^2 = 4$. So the slice of the surface with $y = 0$ is the circle of radius 2 in the xz-plane with centre at the origin. It is sketched in the figure on the left below.
(b) When $y = y_0$, $x^2 + z^2 = 2y_0 + 4$. So, provided $2y_0 + 4 \geq 0$, i.e. provided $y_0 \geq -2$, the slice of the surface with $y = y_0$ is the circle of radius $\sqrt{2y_0 + 4}$ in the plane $y = y_0$ with centre on the y-axis. No point on the surface has $y < -2$. So the surface is a collection of circles stacked sideways along the y-axis, starting with radius 0 when $y = -2$ and with the radius increasing as y increases, i.e. as you move to the right. The slice of the surface in the yz-plane, i.e. with $x = 0$, is the rightward opening parabola $y = z^2/2 - 2$. The surface is sketched in the figure on the right below.