1. Drop a perpendicular from the point \((5, 4, 3)\) to the line \(L\) which passes through the points \((2, 0, 5)\) and \((2, 2, -1)\). Where does the perpendicular hit \(L\)?

**Solution.** The vector from \((2, 0, 5)\) to \((5, 4, 3)\) is \(\langle 5 - 2, 4 - 0, 3 - 5 \rangle = \langle 3, 4, -2 \rangle\). The vector \(\langle 2 - 2, 2 - 0, -1 - 5 \rangle = \langle 0, 2, -6 \rangle\) lies in the line. The projection of \(\langle 3, 4, -2 \rangle\) on \(\langle 0, 2, -6 \rangle\) is

\[
\frac{\langle 3, 4, -2 \rangle \cdot \langle 0, 2, -6 \rangle}{|\langle 0, 2, -6 \rangle|^2} \langle 0, 2, -6 \rangle = \frac{0 + 40}{40} \langle 0, 2, -6 \rangle = \langle 0, 1, -3 \rangle
\]

The perpendicular hits \(L\) at \((2 + 0, 0 + 1, 5 - 3) = \langle 2, 1, 2 \rangle\).

![Perpendicular Diagram](image)

2. For some number \(a\), the points \((1, 1, 0), (2, 3, 1), (2, 1, 2), (3, a, 3)\) are the vertices of a parallelogram.

(a) What is \(a\)? Include your reasoning.
(b) What is the area of the parallelogram?

**Solution.** (a) For the points to be the vertices of a parallelogram, the vector from \((1, 1, 0)\) to \((2, 3, 1)\), namely \(\langle 2 - 1, 3 - 1, 1 - 0 \rangle = \langle 1, 2, 1 \rangle\), must be the same as the vector from \((2, 1, 2)\) to \((3, a, 3)\), namely \(\langle 3 - 2, a - 1, 3 - 2 \rangle = \langle 1, a - 1, 1 \rangle\). So \(a = 3\).

(b) The vector from \((1, 1, 0)\) to \((2, 3, 1)\), namely \(\langle 2 - 1, 3 - 1, 1 - 0 \rangle = \langle 1, 2, 1 \rangle\), and the vector from \((1, 1, 0)\) to \((2, 1, 2)\), namely \(\langle 2 - 1, 1 - 1, 2 - 0 \rangle = \langle 1, 0, 2 \rangle\) form two sides of the parallelogram and

\[
\langle 1, 2, 1 \rangle \times \langle 1, 0, 2 \rangle = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\
1 & 2 & 1 \\
1 & 0 & 2 \\
\end{bmatrix} = \hat{i}(2 \times 2 - 1 \times 0) - \hat{j}(1 \times 2 - 1 \times 1) + \hat{k}(1 \times 0 - 2 \times 1) = 4\hat{i} - \hat{j} - 2\hat{k}
\]

So the area of the parallelogram is \(|4\hat{i} - \hat{j} + 2\hat{k}| = \sqrt{16 + 1 + 4} = \sqrt{21}\).

3. Consider the surface \(x^2 - 2y + z^2 = 4\).

(a) Describe and sketch the slice of this surface with \(y = 0\).
(b) Sketch the surface. Give the reasoning you used to arrive at your sketch.
Solution. (a) When $y = 0$, $x^2 - 2y + z^2 = x^2 + z^2 = 4$. So the slice of the surface with $y = 0$ is the circle of radius 2 in the $xz$-plane with centre at the origin. It is sketched in the figure on the left below.

(b) When $y = y_0$, $x^2 + z^2 = 2y_0 + 4$. So, provided $2y_0 + 4 \geq 0$, i.e. provided $y_0 \geq -2$, the slice of the surface with $y = y_0$ is the circle of radius $\sqrt{2y_0 + 4}$ in the plane $y = y_0$ with centre on the $y$-axis. No point on the surface has $y < -2$. So the surface is a collection of circles stacked sideways along the $y$-axis, starting with radius 0 when $y = -2$ and with the radius increasing as $y$ increases, i.e. as you move to the right. The slice of the surface in the $yz$-plane, i.e. with $x = 0$, is the rightward opening parabola $y = z^2/2 - 2$. The surface is sketched in the figure on the right below.