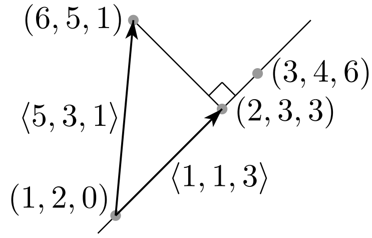
Math 200 Test 1 Solutions **September 20, 2019**

1. Drop a perpendicular from the point $(6, 5, 1)$ to the line L which passes through the points $(1, 2, 0)$ and $(3, 4, 6)$. Where does the perpendicular hit L?

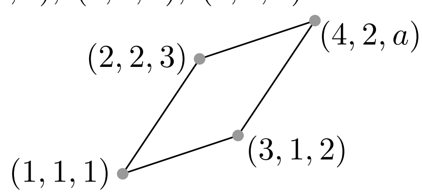
   Solution. The vector from $(1, 2, 0)$ to $(6, 5, 1)$ is $\langle 6-1, 5-2, 1-0\rangle = \langle 5, 3, 1\rangle$. The vector $\langle 3-1, 4-2, 6-0\rangle = \langle 2, 2, 6\rangle$ lies in the line. The projection of $\langle 5, 3, 1\rangle$ on $\langle 2, 2, 6\rangle$ is

$$\frac{\langle 5,3,1\rangle\cdot\langle 2,2,6\rangle}{|\langle 2,2,6\rangle|^2}\,\langle 2, 2, 6\rangle = \frac{22}{44}\,\langle 2, 2, 6\rangle = \langle 1, 1, 3\rangle$$

   The perpendicular hits L at $(1+1, 2+1, 0+3) = \boxed{(2, 3, 3)}$.



2. For some number a, the points $(1, 1, 1)$, $(3, 1, 2)$, $(2, 2, 3)$, $(4, 2, a)$ are the vertices of a parallelogram.
   (a) What is a? Include your reasoning.
   (b) What is the area of the parallelogram?



   Solution. (a) For the points to be the vertices of a parallelogram, the vector from $(1, 1, 1)$ to $(3, 1, 2)$, namely $\langle 3-1, 1-1, 2-1\rangle = \langle 2, 0, 1\rangle$, must be the same as the vector from $(2, 2, 3)$ to $(4, 2, a)$, namely $\langle 4-2, 2-2, a-3\rangle = \langle 2, 0, a-3\rangle$. So $\boxed{a = 4}$.
   (b) The vector from $(1, 1, 1)$ to $(3, 1, 2)$, namely $\langle 3-1, 1-1, 2-1\rangle = \langle 2, 0, 1\rangle$, and the vector from $(1, 1, 1)$ to $(2, 2, 3)$, namely $\langle 2-1, 2-1, 3-1\rangle = \langle 1, 1, 2\rangle$ form two sides of the parallelogram and

$$\begin{aligned}\langle 2, 0, 1\rangle \times \langle 1, 1, 2\rangle &= \det\begin{bmatrix}\hat{\imath} & \hat{\jmath} & \hat{\mathbf{k}}\\ 2 & 0 & 1\\ 1 & 1 & 2\end{bmatrix}\\ &= \hat{\imath}(0\times 2 - 1\times 1) - \hat{\jmath}(2\times 2 - 1\times 1) + \hat{\mathbf{k}}(2\times 1 - 0\times 1)\\ &= -\hat{\imath} - 3\hat{\jmath} + 2\hat{\mathbf{k}}\end{aligned}$$

   So the area of the parallelogram is $|-\hat{\imath} - 3\hat{\jmath} + 2\hat{\mathbf{k}}| = \sqrt{1+9+4} = \boxed{\sqrt{14}}$.

3. Consider the surface $x^2 - y + z^2 = 1$.
   (a) Describe and sketch the slice of this surface with $y = 0$.
   (b) Sketch the surface. Give the reasoning you used to arrive at your sketch.

   Solution. (a) When $y = 0$, $x^2 - y + z^2 = x^2 + z^2 = 1$. So the slice of the surface with $y = 0$ is the circle of radius 1 in the xz-plane with centre at the origin. It is sketched in the figure on the left below.

(b) When $y = y_0$, $x^2 + z^2 = 1 + y_0$. So, provided $1 + y_0 \geq 0$, i.e. provided $y_0 \geq -1$, the slice of the surface with $y = y_0$ is the circle of radius $\sqrt{1 + y_0}$ in the plane $y = y_0$ with centre on the y-axis. No point on the surface has $y < -1$. So the surface is a collection of circles stacked sideways along the y-axis, starting with radius 0 when $y = -1$ and with the radius increasing as y increases. The slice of the surface in the yz-plane, i.e. with $x = 0$, is the parabola $y = z^2 - 1$. The surface is sketched in the figure on the right below.