The Chain Rule Polar Coordinates Example

Example 6: Find the gradient of a function given in polar coordinates.

The gradient of a function $g(x, y)$ is the vector $\big(g_x(x, y), g_y(x, y)\big)$. In this question we are told that we are given some function $f(r, \theta)$ of the polar coordinates r and θ. We are supposed to convert this function to Cartesian coordinates. This means that we are to consider the function

$$g(x, y) = f\big(r(x, y), \theta(x, y)\big)$$

with

$$r(x, y) = \sqrt{x^2 + y^2} \qquad \theta(x, y) = \arctan \tfrac{y}{x}$$

Then we are to compute the gradient of $g(x, y)$ and express the answer in terms of r and θ.

By the chain rule

$$\tfrac{\partial g}{\partial x} = \tfrac{\partial f}{\partial r}\tfrac{\partial r}{\partial x} + \tfrac{\partial f}{\partial \theta}\tfrac{\partial \theta}{\partial x} \qquad \tfrac{\partial g}{\partial x} = \tfrac{\partial f}{\partial r}\tfrac{\partial r}{\partial x} + \tfrac{\partial f}{\partial \theta}\tfrac{\partial \theta}{\partial x}$$

Our main job is to compute $\frac{\partial r}{\partial x}$, $\frac{\partial \theta}{\partial x}$, $\frac{\partial r}{\partial y}$ and $\frac{\partial \theta}{\partial y}$ and express them in terms of r and θ.

$$\begin{aligned}
\frac{\partial r}{\partial x} &= \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}} &&= \frac{x}{\sqrt{x^2+y^2}} &&= \frac{r\cos\theta}{r} &&= \cos\theta \\
\frac{\partial r}{\partial y} &= \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2}} &&= \frac{y}{\sqrt{x^2+y^2}} &&= \frac{r\sin\theta}{r} &&= \sin\theta \\
\frac{\partial \theta}{\partial x} &= \frac{-y/x^2}{1+(y/x)^2} &&= -\frac{y}{x^2+y^2} &&= -\frac{r\sin\theta}{r^2} &&= -\frac{\sin\theta}{r} \\
\frac{\partial \theta}{\partial y} &= \frac{1/x}{1+(y/x)^2} &&= \frac{x}{x^2+y^2} &&= \frac{r\cos\theta}{r^2} &&= \frac{\cos\theta}{r}
\end{aligned}$$

So

$$\begin{aligned}
g_x(x,y) &= \frac{\partial f}{\partial r}\big(r(x,y),\theta(x,y)\big)\ \cos\theta(x,y) \\
&\qquad - \frac{1}{r(x,y)}\frac{\partial f}{\partial \theta}\big(r(x,y),\theta(x,y)\big)\ \sin\theta(x,y) \\
g_y(x,y) &= \frac{\partial f}{\partial r}\big(r(x,y),\theta(x,y)\big)\ \sin\theta(x,y) \\
&\qquad + \frac{1}{r(x,y)}\frac{\partial f}{\partial \theta}\big(r(x,y),\theta(x,y)\big)\ \cos\theta(x,y)
\end{aligned}$$

or, with all the arguments omitted,

$$\begin{bmatrix} g_x \\ g_y \end{bmatrix} = \frac{\partial f}{\partial r}\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + \frac{1}{r}\frac{\partial f}{\partial \theta}\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

(For clarity, I've written the vectors as columns, rather than rows.)