The Chain Rule Polar Coordinates Example

Example 6: Find the gradient of a function given in polar coordinates.

The gradient of a function $g(x, y)$ is the vector $(g_x(x, y), g_y(x, y))$. In this question we are told that we are given some function $f(r, \theta)$ of the polar coordinates $r$ and $\theta$. We are supposed to convert this function to Cartesian coordinates. This means that we are to consider the function

$$g(x, y) = f(r(x, y), \theta(x, y))$$

with

$$r(x, y) = \sqrt{x^2 + y^2} \quad \theta(x, y) = \arctan \frac{y}{x}$$

Then we are to compute the gradient of $g(x, y)$ and express the answer in terms of $r$ and $\theta$.

By the chain rule

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial f}{\partial \theta} \frac{\partial \theta}{\partial x}$$

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Our main job is to compute \( \frac{\partial r}{\partial x}, \frac{\partial \theta}{\partial x}, \frac{\partial r}{\partial y} \) and \( \frac{\partial \theta}{\partial y} \) and express them in terms of \( r \) and \( \theta \).

\[
\begin{align*}
\frac{\partial r}{\partial x} &= \frac{1}{2} \frac{2x}{\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}} = \frac{r \cos \theta}{r} = \cos \theta \\
\frac{\partial r}{\partial y} &= \frac{1}{2} \frac{2y}{\sqrt{x^2+y^2}} = \frac{y}{\sqrt{x^2+y^2}} = \frac{r \sin \theta}{r} = \sin \theta \\
\frac{\partial \theta}{\partial x} &= \frac{-y/x^2}{1+(y/x)^2} = -\frac{y}{x^2+y^2} = -\frac{r \sin \theta}{r^2} = -\sin \theta \\
\frac{\partial \theta}{\partial y} &= \frac{1/x}{1+(y/x)^2} = \frac{x}{x^2+y^2} = \frac{r \cos \theta}{r^2} = \frac{\cos \theta}{r}
\end{align*}
\]

So

\[
\begin{align*}
g_x(x, y) &= \frac{\partial f}{\partial r}(r(x, y), \theta(x, y)) \cos \theta(x, y) \\
&\quad - \frac{1}{r(x,y)} \frac{\partial f}{\partial \theta}(r(x, y), \theta(x, y)) \sin \theta(x, y) \\
g_y(x, y) &= \frac{\partial f}{\partial r}(r(x, y), \theta(x, y)) \sin \theta(x, y) \\
&\quad + \frac{1}{r(x,y)} \frac{\partial f}{\partial \theta}(r(x, y), \theta(x, y)) \cos \theta(x, y)
\end{align*}
\]

or, with all the arguments omitted,

\[
\begin{bmatrix} g_x \\ g_y \end{bmatrix} = \frac{\partial f}{\partial r} \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} + \frac{1}{r} \frac{\partial f}{\partial \theta} \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}
\]

(For clarity, I’ve written the vectors as columns, rather than rows.)