Chain Rule Derivation

Let

$$g(s,t) = f\big(x(s,t), y(s,t)\big)$$

By definition,

$$\frac{\partial g}{\partial s}(s,t) = \lim_{h\to 0} \frac{g(s+h,t) - g(s,t)}{h}$$

$$= \lim_{h\to 0} \frac{f\big(x(s+h,t), y(s+h,t)\big) - f\big(x(s,t), y(s,t)\big)}{h}$$

Applying

$$F(s+h) = F(s) + F'(s)h + O\big(h^2\big)$$

with $F(s) = x(s,t)$ and with $F(s) = y(s,t)$, for any fixed t, gives

$$\begin{aligned} x(s+h,t) &= \underbrace{x(s,t)}_{X} + \underbrace{\tfrac{\partial x}{\partial s}(s,t)h + O(h^2)}_{\Delta x} \\ y(s+h,t) &= \underbrace{y(s,t)}_{Y} + \underbrace{\tfrac{\partial y}{\partial s}(s,t)h + O(h^2)}_{\Delta y} \end{aligned} \tag{1}$$

Subbing (1) into

$$\begin{aligned} f(X+\Delta x, Y+\Delta y) = f(X,Y) & \\ &+ \tfrac{\partial f}{\partial x}(X,Y)\Delta x + \tfrac{\partial f}{\partial y}(X,Y)\Delta y \\ &+ O(\Delta x^2 + \Delta y^2) \end{aligned}$$

gives

$$\begin{aligned}
&\tfrac{1}{h}\Big[g(s+h,t)-g(s,t)\Big] \\
&\quad=\tfrac{1}{h}\Big[f\big(x(s+h,t),y(s+h,t)\big)-f\big(x(s,t),y(s,t)\big)\Big] \\
&\quad=\tfrac{1}{h}\Big[f(X+\Delta x,Y+\Delta y)-f(X,Y)\Big] \\
&\quad=\tfrac{1}{h}\Big[\tfrac{\partial f}{\partial x}(X,Y)\Delta x+\tfrac{\partial f}{\partial y}(X,Y)\Delta y+O(\Delta x^2+\Delta y^2)\Big] \\
&\quad=\tfrac{1}{h}\Big[\tfrac{\partial f}{\partial x}\big(x(s,t),y(s,t)\big)\tfrac{\partial x}{\partial s}(s,t)h \\
&\qquad\qquad+\tfrac{\partial f}{\partial y}\big(x(s,t),y(s,t)\big)\tfrac{\partial y}{\partial s}(s,t)h \\
&\qquad\qquad+O(h^2)\Big] \\
&\quad=\tfrac{\partial f}{\partial x}\big(x(s,t),y(s,t)\big)\tfrac{\partial x}{\partial s}(s,t) \\
&\qquad\qquad+\tfrac{\partial f}{\partial y}\big(x(s,t),y(s,t)\big)\tfrac{\partial y}{\partial s}(s,t) \\
&\qquad\qquad+O(h)
\end{aligned}$$

Dividing by h and taking the limit, we get the upper formula in

$$\boxed{\begin{aligned}
\tfrac{\partial g}{\partial s}(s,t)&=\tfrac{\partial f}{\partial x}\big(x(s,t),y(s,t)\big)\tfrac{\partial x}{\partial s}(s,t) \\
&\quad+\tfrac{\partial f}{\partial y}\big(x(s,t),y(s,t)\big)\tfrac{\partial y}{\partial s}(s,t) \\
\tfrac{\partial g}{\partial t}(s,t)&=\tfrac{\partial f}{\partial x}\big(x(s,t),y(s,t)\big)\tfrac{\partial x}{\partial t}(s,t) \\
&\quad+\tfrac{\partial f}{\partial y}\big(x(s,t),y(s,t)\big)\tfrac{\partial y}{\partial t}(s,t)
\end{aligned}}$$

The lower formula is derived similarly.