A Chain Rule 2^{nd} Derivative Example

Example 5: Suppose that $f(x,y)=0$. Find $\frac{d^2y}{dx^2}$.

Again, x and y are not independent variables. Given a value for either x or y, the other is determined by solving $f(x,y)=0$. Since we are asked to find $\frac{d^2y}{dx^2}$, it is y that is to be viewed as a function of x. So $f(x,y)=0$ really means, in this problem, $f\big(x,y(x)\big)=0$ for all x. Differentiating both sides of this equation with respect to x,

$$\begin{aligned}
&f\big(x,y(x)\big)=0 \qquad \text{for all } x\\
\Longrightarrow\ &\tfrac{d}{dx}f\big(x,y(x)\big)=0\\
\Longrightarrow\ &f_x\big(x,y(x)\big)\tfrac{dx}{dx}+f_y\big(x,y(x)\big)\tfrac{dy}{dx}(x)=0\\
\Longrightarrow\ &\frac{dy}{dx}(x)=-\frac{f_x\big(x,y(x)\big)}{f_y\big(x,y(x)\big)}\\
\Longrightarrow\ &\frac{d^2y}{dx^2}(x)=-\frac{d}{dx}\Big[\frac{f_x\big(x,y(x)\big)}{f_y\big(x,y(x)\big)}\Big]\\
=-&\frac{f_y\big(x,y(x)\big)\frac{d}{dx}\big[f_x\big(x,y(x)\big)\big]-f_x\big(x,y(x)\big)\frac{d}{dx}\big[f_y\big(x,y(x)\big)\big]}{f_y\big(x,y(x)\big)^2}
\end{aligned}$$

by the quotient rule. Now it suffices to substitute in $\frac{d}{dx}[f_x\big(x, y(x)\big)]$ and $\frac{d}{dx}[f_y\big(x, y(x)\big)]$. For the former apply the chain rule to $\gamma(x) = \varphi\big(x, y(x)\big)$ with $\varphi(x, y) = f_x\big(x, y\big)$.

$$\begin{aligned}
&\tfrac{d}{dx}[f_x\big(x, y(x)\big)] = \tfrac{d\gamma}{dx}(x) \\
&= \tfrac{\partial\varphi}{\partial x}\big(x, y(x)\big)\tfrac{dx}{dx} + \tfrac{\partial\varphi}{\partial y}\big(x, y(x)\big)\tfrac{dy}{dx}(x) \\
&= f_{xx}\big(x, y(x)\big)\tfrac{dx}{dx} + f_{xy}\big(x, y(x)\big)\tfrac{dy}{dx}(x) \\
&= f_{xx}\big(x, y(x)\big) - f_{xy}\big(x, y(x)\big)\left[\frac{f_x\big(x, y(x)\big)}{f_y\big(x, y(x)\big)}\right]
\end{aligned}$$

Substituting this and

$$\begin{aligned}
&\tfrac{d}{dx}[f_y\big(x, y(x)\big)] = f_{yx}\big(x, y(x)\big)\tfrac{dx}{dx} + f_{yy}\big(x, y(x)\big)\tfrac{dy}{dx}(x) \\
&= f_{yx}\big(x, y(x)\big) - f_{yy}\big(x, y(x)\big)\left[\frac{f_x\big(x, y(x)\big)}{f_y\big(x, y(x)\big)}\right]
\end{aligned}$$

into the right hand side gives the final answer.

Note that the “total derivative” $\frac{d}{dx}f\big(x, y(x)\big)$ is not the same as the partial derivative $f_x\big(x, y(x)\big)$. The former is, by definition, the rate of change with respect to x of $g(x) = f\big(x, y(x)\big)$. Precisely,

$$\begin{aligned}\frac{dg}{dx} &= \lim_{\Delta x\to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{f\big(x+\Delta x, y(x+\Delta x)\big) - f\big(x, y(x)\big)}{\Delta x}\end{aligned}$$

On the other hand

$$\begin{aligned}f_x(x,y) &= \lim_{\Delta x\to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} \\ \Rightarrow f_x\big(x, y(x)\big) &= \lim_{\Delta x\to 0} \frac{f\big(x+\Delta x, y(x)\big) - f\big(x, y(x)\big)}{\Delta x}\end{aligned}$$

The two right hand sides are not the same. In the first, as Δx varies the value of y that is substituted into the first $f(\cdots)$, namely $y(x+\Delta x)$ varies. In the second, the corresponding value of y is $y(x)$ and is independent of Δx.