MATHEMATICS 200 December 2014 Final Exam Solutions

1. Suppose that \( f(x, y, z) \) is a function of three variables and let \( u = \frac{1}{\sqrt{6}} \langle 1, 1, 2 \rangle \) and \( v = \frac{1}{\sqrt{3}} \langle 1, -1, -1 \rangle \) and \( w = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle \). Suppose that at a point \((a, b, c)\),

\[
D_u f = 0 \\
D_v f = 0 \\
D_w f = 4
\]

Find \( \nabla f \) at \((a, b, c)\).

**Solution.** Write \( \nabla f(a, b, c) = \langle F, G, H \rangle \). We are told that

\[
D_u f = \frac{1}{\sqrt{6}} \langle 1, 1, 2 \rangle \cdot \langle F, G, H \rangle = 0 \\
D_v f = \frac{1}{\sqrt{3}} \langle 1, -1, -1 \rangle \cdot \langle F, G, H \rangle = 0 \\
D_w f = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle \cdot \langle F, G, H \rangle = 4
\]

so that

\[
F + G + 2H = 0 \quad \text{(E1)} \\
F - G - H = 0 \quad \text{(E2)} \\
F + G + H = 4\sqrt{3} \quad \text{(E3)}
\]

Adding (E2) and (E3) gives \( 2F = 4\sqrt{3} \) or \( F = 2\sqrt{3} \). Substituting \( F = 2\sqrt{3} \) into (E1) and (E2) gives

\[
G + 2H = -2\sqrt{3} \quad \text{(E1)} \\
-G - H = -2\sqrt{3} \quad \text{(E2)}
\]

Adding (E1) and (E2) gives \( H = -4\sqrt{3} \) and substituting \( H = -4\sqrt{3} \) back into (E2) gives \( G = 6\sqrt{3} \). All together

\[
\nabla f(a, b, c) = \sqrt{3} \langle 2, 6, -4 \rangle
\]

2. Let \( f(u, v) \) be a differentiable function of two variables, and let \( z \) be a differentiable function of \( x \) and \( y \) defined implicitly by \( f(xz, yz) = 0 \). Show that

\[
x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = -z
\]

**Solution.** We are told that the function \( z(x, y) \) obeys

\[
f(x z(x, y), y z(x, y)) = 0
\]
for all \( x \) and \( y \). Differentiating this equation with respect to \( x \) and with respect to \( y \) gives, by the chain rule,

\[
\begin{align*}
   f_u(z(x,y), y z(x,y)) \left[ z(x,y) + x z_x(x,y) \right] + f_v(z(x,y), y z(x,y)) y z_x(x,y) &= 0 \\
   f_u(z(x,y), y z(x,y)) x z_y(x,y) + f_v(z(x,y), y z(x,y)) \left[ z(x,y) + y z_y(x,y) \right] &= 0
\end{align*}
\]

or, leaving out the arguments,

\[
\begin{align*}
   f_u(z + x z_x) + f_v y z_x &= 0 \\
   f_u x z_y + f_v [z + y z_y] &= 0
\end{align*}
\]

Solving the first equation for \( z_x \) and the second for \( z_y \) gives

\[
\begin{align*}
   z_x &= -\frac{z f_u}{x f_u + y f_v} \\
   z_y &= -\frac{z f_v}{x f_u + y f_v}
\end{align*}
\]

so that

\[
\begin{align*}
   x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} &= -\frac{z f_u}{x f_u + y f_v} - \frac{y z f_v}{x f_u + y f_v} = -\frac{z (x f_u + y f_v)}{x f_u + y f_v} = -z
\end{align*}
\]

as desired.

Remark: This is of course under the assumption that \( x f_u + y f_v \) is nonzero. That is equivalent, by the chain rule, to the assumption that \( \frac{\partial}{\partial z} [f(xz, yz)] \) is non zero. That, in turn, is almost, but not quite, equivalent to the statement that \( f(xz, yz) = 0 \) is soluble for \( z \) as a function of \( x \) and \( y \).

3. Let \( z = f(x, y) \) be given implicitly by

\[
e^z + yz = x + y.
\]

(a) Find the differential \( dz \).

(b) Use linear approximation at the point \((1, 0)\) to approximate \( f(0.99, 0.01) \).

Solution. By definition, the differential at \( x = a, \ y = b \) is

\[
f_x(a, b) \, dx + f_y(a, b) \, dy
\]

so we have to determine the partial derivatives \( f_x(a, b) \) and \( f_y(a, b) \). We are told that

\[
e^f(x,y) + y f(x,y) = x + y
\]
for all \( x \) and \( y \). Differentiating this equation with respect to \( x \) and with respect to \( y \) gives, by the chain rule,

\[
e^{f(x,y)} f_x(x, y) + y f_x(x, y) = 1
\]

\[
e^{f(x,y)} f_y(x, y) + f(x, y) + y f_y(x, y) = 1
\]

Solving the first equation for \( f_x \) and the second for \( f_y \) gives

\[
f_x(x, y) = \frac{1}{e^{f(x,y)} + y}
\]

\[
f_y(x, y) = \frac{1 - f(x, y)}{e^{f(x,y)} + y}
\]

So the differential at \( x = a, \ y = b \) is

\[
\frac{dx}{e^{f(a,b)} + b} + \frac{1 - f(a,b)}{e^{f(a,b)} + b} \, dy
\]

Since we can’t solve explicitly for \( f(a,b) \) for general \( a \) and \( b \). There’s not much more that we can do with this.

(b) In particular, when \( a = 1 \) and \( b = 0 \), we have

\[
e^{f(1,0)} + 0 f(1,0) = 1 + 0 \implies e^{f(1,0)} = 1 \implies f(1,0) = 0
\]

and the linear approximation simplifies to

\[
f(1 + dx, dy) \approx f(1,0) + \frac{dx}{e^{f(1,0)} + 0} + \frac{1 - f(1,0)}{e^{f(1,0)} + 0} \, dy = dx + dy
\]

Choosing \( dx = -0.01 \) and \( dy = 0.01 \), we have

\[
f(0.99, 0.01) \approx -0.01 + 0.01 = 0
\]

4. Let \( S \) be the surface \( z = x^2 + 2y^2 + 2y - 1 \). Find all points \( P(x_0, y_0, z_0) \) on \( S \) with \( x_0 \neq 0 \) such that the normal line at \( P \) contains the origin \( (0,0,0) \).

**Solution.** The equation of \( S \) is of the form \( G(x, y, z) = x^2 + 2y^2 + 2y - z = 1 \). So one normal vector to \( S \) at the point \( (x_0, y_0, z_0) \) is

\[
\nabla G(x_0, y_0, z_0) = 2x_0 \hat{i} + (4y_0 + 2) \hat{j} - \hat{k}
\]

and the normal line to \( S \) at \( (x_0, y_0, z_0) \) is

\[
(x, y, z) = (x_0, y_0, z_0) + t (2x_0, 4y_0 + 2, -1)
\]

For this normal line to pass through the origin, there must be a \( t \) with

\[
(0, 0, 0) = (x_0, y_0, z_0) + t (2x_0, 4y_0 + 2, -1)
\]
or

\[ x_0 + 2x_0 t = 0 \quad \text{(E1)} \]
\[ y_0 + (4y_0 + 2)t = 0 \quad \text{(E2)} \]
\[ z_0 - t = 0 \quad \text{(E3)} \]

Equation (E3) forces \( t = z_0 \). Substituting this into equations (E1) and (E2) gives

\[ x_0(1 + 2z_0) = 0 \quad \text{(E1)} \]
\[ y_0 + (4y_0 + 2)z_0 = 0 \quad \text{(E2)} \]

The question specifies that \( x_0 \neq 0 \), so (E1) forces \( z_0 = -\frac{1}{2} \). Substituting \( z_0 = -\frac{1}{2} \) into (E2) gives

\[ -y_0 - 1 = 0 \implies y_0 = -1 \]

Finally \( x_0 \) is determined by the requirement that \((x_0, y_0, z_0)\) must lie on \( S \) and so must obey

\[ z_0 = x_0^2 + 2y_0^2 + 2y_0 - 1 \implies -\frac{1}{2} = x_0^2 + 2(-1)^2 + 2(-1) - 1 \implies x_0^2 = \frac{1}{2} \]

So the allowed points \( P \) are \((\frac{1}{\sqrt{2}}, -1, -\frac{1}{2})\) and \((-\frac{1}{\sqrt{2}}, -1, -\frac{1}{2})\).

5. Let \( f(x, y) = xy(x + y - 3) \).

(a) Find all critical points of \( f \), and classify each one as a local maximum, a local minimum, or saddle point.

(b) Find the location and value of the absolute maximum and minimum of \( f \) on the triangular region \( x \geq 0, y \geq 0, x + y \leq 8 \).

**Solution.** (a) Thinking a little way ahead, to find the critical points we will need the gradient of \( f \) and to apply the second derivative test of Theorem 2.9.16 in the CLP–III text we will need all second order partial derivatives. So we need all partial derivatives of \( f \) up to order two. Here they are.

\[
\begin{align*}
  f &= xy(x + y - 3) \\
  f_x &= 2xy + y^2 - 3y & f_{xx} &= 2y & f_{xy} &= 2x + 2y - 3 \\
  f_y &= x^2 + 2xy - 3x & f_{yy} &= 2x & f_{yx} &= 2x + 2y - 3 \\
\end{align*}
\]

(Of course, \( f_{xy} \) and \( f_{yx} \) have to be the same. It is still useful to compute both, as a way to catch some mechanical errors.)

The critical points are the solutions of

\[ f_x = y(2x + y - 3) = 0 \quad f_y = x(x + 2y - 3) = 0 \]

The first equation is satisfied if at least one of \( y = 0, y = 3 - 2x \) are satisfied.
• If $y = 0$, the second equation reduces to $x(x - 3) = 0$, which is satisfied if either $x = 0$ or $x = 3$.
• If $y = 3 - 2x$, the second equation reduces to $x(x + 6 - 4x - 3) = x(3 - 3x) = 0$ which is satisfied if $x = 0$ or $x = 1$.

So there are four critical points: $(0,0), (3,0), (0,3), (1,1)$.

The classification is

<table>
<thead>
<tr>
<th>critical point</th>
<th>$f_{xx}f_{yy} - f_{xy}^2</th>
<th>f_{xx}</th>
<th>type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0,0)$</td>
<td>$0 \times 0 - (-3)^2 &lt; 0$</td>
<td>$0$</td>
<td>saddle point</td>
</tr>
<tr>
<td>$(3,0)$</td>
<td>$0 \times 6 - (3)^2 &lt; 0$</td>
<td>$0$</td>
<td>saddle point</td>
</tr>
<tr>
<td>$(0,3)$</td>
<td>$6 \times 0 - (3)^2 &lt; 0$</td>
<td>$0$</td>
<td>saddle point</td>
</tr>
<tr>
<td>$(1,1)$</td>
<td>$2 \times 2 - (1)^2 &gt; 0$</td>
<td>$2$</td>
<td>local min</td>
</tr>
</tbody>
</table>

(b) The absolute max and min can occur either in the interior of the triangle or on the boundary of the triangle. The boundary of the triangle consists of the three line segments.

$L_1 = \{(x,y) \mid x = 0, \ 0 \leq y \leq 8 \}$
$L_2 = \{(x,y) \mid y = 0, \ 0 \leq x \leq 8 \}$
$L_3 = \{(x,y) \mid x + y = 8, \ 0 \leq x \leq 8 \}$

• Any absolute max or min in the interior of the triangle must also be a local max or min and, since there are no singular points, must also be a critical point of $f$. We found all of the critical points of $f$ in part (a). Only one of them, namely $(1,1)$ is in the interior of the triangle. (The other three critical points are all on the boundary of the triangle.) We have $f(1,1) = -1$.
• At each point of $L_1$ we have $x = 0$ and so $f(x,y) = 0$.
• At each point of $L_2$ we have $y = 0$ and so $f(x,y) = 0$.
• At each point of $L_3$ we have $f(x,y) = x(8-x)(5) = 40x - 5x^2 = 5[8x - x^2]$ with $0 \leq x \leq 8$. As $\frac{d}{dx}(40x - 5x^2) = 40 - 10x$, the max and min of $40x - 5x^2$ on $0 \leq x \leq 8$ must be one of $5[8x - x^2]_{x=0} = 0$ or $5[8x - x^2]_{x=8} = 0$ or $5[8x - x^2]_{x=4} = 80$.

So all together, we have the following candidates for max and min, with the max and min indicated.

<table>
<thead>
<tr>
<th>point(s)</th>
<th>$(1,1)$</th>
<th>$L_1$</th>
<th>$L_2$</th>
<th>$(0,8)$</th>
<th>$(8,0)$</th>
<th>$(4,4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>value of $f$</td>
<td>$-1$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$80$</td>
</tr>
<tr>
<td>min</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>max</td>
</tr>
</tbody>
</table>
6. In the $xy$–plane, the disk $x^2 + y^2 \leq 2x$ is cut into 2 pieces by the line $y = x$. Let $D$ be the larger piece.

(a) Sketch $D$ including an accurate description of the center and radius of the given disk. Then describe $D$ in polar coordinates $(r, \theta)$.

(b) Find the volume of the solid below $z = \sqrt{x^2 + y^2}$ and above $D$.

**Solution.** (a) The inequality $x^2 + y^2 \leq 2x$ is equivalent to $(x - 1)^2 + y^2 \leq 1$ and says that $(x, y)$ is to be inside the disk of radius 1 centred on $(1, 0)$. Here is a sketch.

In polar coordinates, $x = r \cos \theta, y = r \sin \theta$ so that the line $y = x$ is $\theta = \pi/4$ and the circle $x^2 + y^2 = 2x$ is

$$r^2 = 2r \cos \theta \quad \text{or} \quad r = 2 \cos \theta$$

Consequently

$$D = \{ (r \cos \theta, r \sin \theta) \mid -\pi/2 \leq \theta \leq \pi/4, \ 0 \leq r \leq 2 \cos \theta \}$$

(b) The solid has height $z = r$ above the point in $D$ with polar coordinates $r, \theta$. So the

$$\text{Volume} = \iiint_D r \, dA = \iiint_D r^2 \, dr \, d\theta = \int_{-\pi/2}^{\pi/4} \, d\theta \int_0^{2 \cos \theta} r^2 \, dr$$

$$= \frac{8}{3} \int_{-\pi/2}^{\pi/4} \, d\theta \cos^3 \theta = \frac{8}{3} \int_{-\pi/2}^{\pi/4} \, d\theta \cos \theta [1 - \sin^2 \theta]$$

$$= \frac{8}{3} \left[ \frac{\sin \theta - \sin^3 \theta}{3} \right]_{-\pi/2}^{\pi/4}$$

$$= \frac{8}{3} \left[ \left( \frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}} \right) - \left( -1 + \frac{1}{3} \right) \right]$$

$$= \frac{40}{18\sqrt{2}} + \frac{16}{9}$$

7. The density of hydrogen gas in a region of space is given by the formula

$$\rho(x, y, z) = \frac{z + 2x^2}{1 + x^2 + y^2}$$
(a) At \((1, 0, -1)\), in which direction is the density of hydrogen increasing most rapidly?

(b) You are in a spacecraft at the origin. Suppose the spacecraft flies in the direction of \(\langle 0, 0, 1 \rangle\). It has a disc of radius 1, centred on the spacecraft and deployed perpendicular to the direction of travel, to catch hydrogen. How much hydrogen has been collected by the time that the spacecraft has traveled a distance 2? [You may use the fact that \(\int_0^{2\pi} \cos^2 \theta \, d\theta = \pi\).]

Solution. (a) The direction of maximum rate of increase is \(\nabla \rho(1, 0, -1)\). As

\[
\frac{\partial \rho}{\partial x}(x, y, z) = \frac{4x}{1 + x^2 + y^2} - \frac{2x(z + 2x^2)}{(1 + x^2 + y^2)^2} \quad \frac{\partial \rho}{\partial x}(1, 0, -1) = \frac{4}{2} - \frac{2(-1 + 2)}{(2)^2} = \frac{3}{2}
\]

\[
\frac{\partial \rho}{\partial y}(x, y, z) = -\frac{2y(z + 2x^2)}{(1 + x^2 + y^2)^2} \quad \frac{\partial \rho}{\partial y}(1, 0, -1) = 0
\]

\[
\frac{\partial \rho}{\partial z}(x, y, z) = \frac{1}{1 + x^2 + y^2} \quad \frac{\partial \rho}{\partial z}(1, 0, -1) = \frac{1}{2}
\]

So \(\nabla \rho(1, 0, -1) = \frac{1}{2}(3, 0, 1)\). The unit vector in this direction is \(\frac{1}{\sqrt{10}}(3, 0, 1)\).

(b) The region swept by the spacecraft is, in cylindrical coordinates,

\[
V = \{ (r \cos \theta, r \sin \theta, z) \mid 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1, 0 \leq z \leq 2 \}
\]

and the amount of hydrogen collected is

\[
\iiint_V \rho \, dV = \iiint_V \frac{z + 2r^2 \cos^2 \theta}{1 + r^2} r \, dr \, d\theta \, dz
\]

\[
= \int_0^2 \int_0^{2\pi} \int_0^1 \frac{zr + (2 \cos^2 \theta)r^3}{1 + r^2} \frac{1}{1 + r^2} \, dr \, d\theta \, dz
\]

\[
= \int_0^2 \int_0^{2\pi} \int_0^1 \frac{zr}{1 + r^2} + 2r \cos^2 \theta - \cos^2 \theta \frac{2r}{1 + r^2} \left[ z \frac{r}{1 + r^2} + 2r \cos^2 \theta - \cos^2 \theta \frac{2r}{1 + r^2} \right]
\]

since \(\frac{r^3}{1 + r^2} = \frac{r - r^3}{1 + r^2} = r - \frac{r}{1 + r^2}\)

\[
= \int_0^2 \int_0^{2\pi} \frac{z}{2} \ln(1 + r^2) + r^2 \cos^2 \theta - \ln(1 + r^2) \cos^2 \theta \left[ \frac{\ln \frac{r}{2}}{2} + \cos^2 \theta - \ln(2) \cos^2 \theta \right]
\]

\[
= \int_0^2 \int_0^{2\pi} \left[ \frac{\ln \frac{r}{2}}{2} + \cos^2 \theta - \ln(2) \cos^2 \theta \right]
\]

\[
= \int_0^2 \int_0^{2\pi} \left[ (\pi \ln 2)z + \pi - \pi \ln 2 \right]
\]

\[
= 2\pi \ln 2 + 2\pi - 2\pi \ln 2
\]

\[
= 2\pi
\]
8. Consider the iterated integral

\[ I = \int_{-a}^{0} \int_{\sqrt{a^2-x^2}}^{0} \int_{0}^{\sqrt{a^2-x^2-y^2}} (x^2 + y^2 + z^2)^{2014} \, dz \, dy \, dx \]

where \( a \) is a positive constant.

(a) Write \( I \) as an iterated integral in cylindrical coordinates.

(b) Write \( I \) as an iterated integral in spherical coordinates.

(c) Evaluate \( I \) using whatever method you prefer.

**Solution.** The main step is to figure out what the domain of integration looks like.

- The outside integral says that \( x \) runs from \(-a\) to 0.
- The middle integrals says that, for each \( x \) in that range, \( y \) runs from \(-\sqrt{a^2-x^2}\) to 0. We can rewrite \( y = -\sqrt{a^2-x^2} \) in the more familiar form \( x^2 + y^2 = a^2, \ y \leq 0 \). So \((x, y)\) runs over the third quadrant part of the disk of radius \( a \), centred on the origin.

- Finally, the inside integral says that, for each \((x, y)\) in the quarter disk, \( z \) runs from 0 to \( \sqrt{a^2-x^2-y^2} \). We can also rewrite \( z = \sqrt{a^2-x^2-y^2} \) in the more familiar form \( x^2 + y^2 + z^2 = a^2, \ z \geq 0 \).

So the domain of integration is the part of the interior of the sphere of radius \( a \), centred on the origin, that lies in the octant \( x \leq 0, \ y \leq 0, \ z \geq 0 \).

\[
V = \{ (x,y,z) \mid -a \leq x \leq 0, \ -\sqrt{a^2-x^2} \leq y \leq 0, \ 0 \leq z \leq \sqrt{a^2-x^2-y^2} \} \\
= \{ (x,y,z) \mid x^2 + y^2 + z^2 \leq a^2, \ x \leq 0, \ y \leq 0, \ z \geq 0 \} 
\]
(a) Note that, in $V$, $(x, y)$ is restricted to the third quadrant, which in cylindrical coordinates is $\pi \leq \theta \leq \frac{3\pi}{2}$. So, in cylindrical coordinates,

$$V = \left\{(r \cos \theta, r \sin \theta, z) \mid r^2 + z^2 \leq a^2, \pi \leq \theta \leq \frac{3\pi}{2}, z \geq 0 \right\}$$

$$= \left\{(r \cos \theta, r \sin \theta, z) \mid 0 \leq z \leq a, \pi \leq \theta \leq \frac{3\pi}{2}, 0 \leq r \leq \sqrt{a^2 - z^2} \right\}$$

and

$$I = \iiint_V (x^2 + y^2 + z^2)^{2014} \, dV = \iiint_V (r^2 + z^2)^{2014} \, r \, dr \, d\theta \, dz$$

$$= \int_0^a \int_\pi^{3\pi/2} \int_0^{\sqrt{a^2 - z^2}} r \, dr \, d\theta \, dz (r^2 + z^2)^{2014}$$

(b) The spherical coordinate $\varphi$ runs from 0 (when the radius vector is along the positive $z$-axis) to $\pi/2$ (when the radius vector lies in the $xy$-plane) so that

$$I = \iiint_V (x^2 + y^2 + z^2)^{2014} \, dV = \iiint_V \rho^{2 \times 2014} \rho^2 \sin \varphi \, d\rho \, d\theta \, d\varphi$$

$$= \int_0^{\pi/2} d\varphi \int_\pi^{3\pi/2} d\theta \int_0^a d\rho \rho^{4030} \sin \varphi$$

(c) Using the spherical coordinate version

$$I = \int_0^{\pi/2} d\varphi \int_\pi^{3\pi/2} d\theta \int_0^a d\rho \rho^{4030} \sin \varphi$$

$$= a^{4031} \int_0^{\pi/2} d\varphi \int_\pi^{3\pi/2} d\theta \sin \varphi$$

$$= a^{4031} \pi \int_0^{\pi/2} d\varphi \sin \varphi$$

$$= a^{4031} \pi \cdot \frac{8062}{8062}$$

$$= \frac{a^{4031} \pi}{8062}$$
9. Let $E$ be the region bounded by $z = 2x$, $z = y^2$, and $x = 3$. The triple integral $\iiint f(x, y, z) \, dV$ can be expressed as an iterated integral in the following three orders of integration. Fill in the limits of integration in each case. No explanation required.

\[
\int_{y=\ldots}^{y=\ldots} \int_{x=\ldots}^{x=\ldots} \int_{z=\ldots}^{z=\ldots} f(x, y, z) \, dz \, dx \, dy
\]

\[
\int_{y=\ldots}^{y=\ldots} \int_{z=\ldots}^{z=\ldots} \int_{x=\ldots}^{x=\ldots} f(x, y, z) \, dx \, dz \, dy
\]

\[
\int_{z=\ldots}^{z=\ldots} \int_{x=\ldots}^{x=\ldots} \int_{y=\ldots}^{y=\ldots} f(x, y, z) \, dy \, dx \, dz
\]

Solution. The hard part of this problem is figuring out what $E$ looks like. First here are separate sketches of the plane $x = 3$ and the plane $z = 2x$ followed by a sketch of the two planes together.

Next for the parabolic cylinder $z = y^2$. It is a bunch of parabolas $z = y^2$ stacked side by side along the $x$–axis. Here is a sketch of the part of $z = y^2$ in the first octant.

Finally, here is a sketch of the part of $E$ in the first octant. $E$ does have a second half gotten from the sketch by reflecting it in the $xz$–plane, i.e. by replacing $y \rightarrow -y$. 
\[ E = \{ (x, y, z) \mid x \leq 3, -\sqrt{6} \leq y \leq \sqrt{6}, y^2 \leq z \leq 2x \} \]

Order \( dz\, dx\, dy \): On \( E \), \( y \) runs from \(-\sqrt{6}\) to \(\sqrt{6}\). For each fixed \( y \) in this range \((x, z)\) runs over \( E_y = \{ (x, z) \mid x \leq 3, y^2 \leq z \leq 2x \} \). Here is a sketch of \( E_y \).

From the sketch
\[ E_y = \{ (x, z) \mid y^2/2 \leq x \leq 3, y^2 \leq z \leq 2x \} \]

and the integral is
\[
\int_{y=-\sqrt{6}}^{y=\sqrt{6}} \int_{x=y^2/2}^{x=3} \int_{z=y^2}^{z=2x} f(x, y, z) \, dz\, dx\, dy
\]

\(^{1}\)The question doesn’t specify on which side of the three surfaces \( E \) lies. When in doubt take the finite region bounded by the given surfaces. That’s what we have done.
Order $dx \, dz \, dy$: Also from the sketch of $E_y$ above

$$E_y = \{ (x, z) \mid y^2 \leq z \leq 6, \ z/2 \leq x \leq 3 \}$$

and the integral is

$$\int_{y=-\sqrt{6}}^{y=\sqrt{6}} \int_{z=y^2}^{z=6} \int_{x=z/2}^{x=3} f(x, y, z) \, dx \, dz \, dy$$

Order $dy \, dx \, dz$: From the sketch of the part of $E$ in the first octant, we see that, on $E$, $z$ runs from 0 to 6. For each fixed $z$ in this range $(x, y)$ runs over

$$E_z = \{ (x, y) \mid x \leq 3, \ -\sqrt{6} \leq y \leq \sqrt{6}, \ y^2 \leq z \leq 2x \}$$

$$= \{ (x, y) \mid x/2 \leq x \leq 3, \ y^2 \leq z \}$$

$$= \{ (x, y) \mid z/2 \leq x \leq 3, \ -\sqrt{z} \leq y \leq \sqrt{z} \}$$

So the integral is

$$\int_{z=0}^{z=6} \int_{x=z/2}^{x=3} \int_{y=-\sqrt{z}}^{y=\sqrt{z}} f(x, y, z) \, dy \, dx \, dz$$