MATHEMATICS 200 April 2014 Final Exam Solutions

1. Consider two planes $W_1, W_2$, and a line $M$ defined by:

\[ W_1 : -2x + y + z = 7, \quad W_2 : -x + 3y + 3z = 6, \quad M : \frac{x}{2} = \frac{2y - 4}{4} = z + 5. \]

(a) Find a parametric equation of the line of intersection $L$ of $W_1$ and $W_2$.

(b) Find the distance from $L$ to $M$.

(c) Find the area of the parallelogram on $W_2 (-x + 3y + 3z = 6)$ defined by $0 \leq x \leq 3, 0 \leq y \leq 2$.

Solution. (a) Let’s use $z$ as the parameter and rename it to $t$. That is, $z = t$. Subtracting 2 times the $W_2$ equation from the $W_1$ equation gives

\[-5y - 5z = -5 \implies y = 1 - z = 1 - t\]

Substituting the result into the equation for $W_2$ gives

\[-x + 3(1 - t) + 3t = 6 \implies x = -3\]

So a parametric equation is

\[(x, y, z) = (-3, 1, 0) + t \langle 0, -1, 1 \rangle\]

(b) Solution 1

We can also parametrize $M$ using $z = t$:

\[x = 2z + 10 = 2t + 10, \quad y = 2z + 12 = 2t + 12 \implies (x, y, z) = (10, 12, 0) + t \langle 2, 2, 1 \rangle\]

So one point on $M$ is $(10, 12, 0)$ and one point on $L$ is $(-3, 1, 0)$ and

\[v = \langle (-3) - (10), (1) - (12), 0 - 0 \rangle = \langle -13, -11, 0 \rangle\]

is one vector from a point on $M$ to a point on $L$.

The direction vectors of $L$ and $M$ are $\langle 0, -1, 1 \rangle$ and $\langle 2, 2, 1 \rangle$, respectively. The vector

\[n = \langle 0, -1, 1 \rangle \times \langle 2, 2, 1 \rangle = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -1 & 1 \\ 2 & 2 & 1 \end{bmatrix} = \langle -3, 2, 2 \rangle\]

is then perpendicular to both $L$ and $M$.

The distance from $L$ to $M$ is then the length of the projection of $v$ on $n$, which is

\[\frac{|v \cdot n|}{|n|} = \frac{|39 - 22 + 0|}{\sqrt{9 + 4 + 4}} = \sqrt{17}\]
(b) **Solution 2** We can also parametrize $M$ using $z = s$:

$$x = 2z + 10 = 2s + 10, \quad y = 2z + 12 = 2s + 12 \quad \implies \quad (x, y, z) = (10, 12, 0) + s \langle 2, 2, 1 \rangle$$

The vector from the point $(-3, 1, 0) + t \langle 0, -1, 1 \rangle$ on $L$ to the point $(10, 12, 0) + s \langle 2, 2, 1 \rangle$ on $M$ is

$$\langle 13 + 2s, 11 + 2s + t, s - t \rangle$$

So the distance from the point $(-3, 1, 0) + t \langle 0, -1, 1 \rangle$ on $L$ to the point $(10, 12, 0) + s \langle 2, 2, 1 \rangle$ on $M$ is the square root of

$$D(s, t) = (13 + 2s)^2 + (11 + 2s + t)^2 + (s - t)^2$$

That distance is minimized when

$$0 = \frac{\partial D}{\partial s} = 4(13 + 2s) + 4(11 + 2s + t) + 2(s - t)$$

$$0 = \frac{\partial D}{\partial t} = 2(11 + 2s + t) - 2(s - t)$$

Cleaning up those equations gives

$$18s + 2t = -96$$

$$2s + 4t = -22$$

or

$$9s + t = -48 \quad \text{(E1)}$$

$$s + 2t = -11 \quad \text{(E2)}$$

Subtracting (E2) from twice (E1) gives

$$17s = -85 \implies s = -5$$

Substituting that into (E2) gives

$$2t = -11 + 5 \implies t = -3$$

Note that

$$13 + 2s = 3$$

$$11 + 2s + t = -2$$

$$s - t = -2$$

So the distance is

$$\sqrt{D(-5, -3)} = \sqrt{3^2 + (-2)^2 + (-2)^2} = \sqrt{17}$$
(c) Note that

- the point on $W_2$ with $x = 0, y = 0$ obeys $-0 + 3(0) + 3z = 6$ and so has $z = 2$
- the point on $W_2$ with $x = 0, y = 2$ obeys $-0 + 3(2) + 3z = 6$ and so has $z = 0$
- the point on $W_2$ with $x = 3, y = 0$ obeys $-3 + 3(0) + 3z = 6$ and so has $z = 3$
- the point on $W_2$ with $x = 3, y = 2$ obeys $-3 + 3(2) + 3z = 6$ and so has $z = 1$

So the four corners of the parallelogram are $(0, 0, 2), (0, 2, 0), (3, 0, 3)$ and $(3, 2, 1)$. The vectors

\[
\mathbf{d}_1 = \langle 0 - 0, 2 - 0, 0 - 2 \rangle = \langle 0, 2, -2 \rangle
\]

\[
\mathbf{d}_2 = \langle 3 - 0, 0 - 0, 3 - 2 \rangle = \langle 3, 0, 1 \rangle
\]

form two sides of the parallelogram. So the area of the parallelogram is

\[
|\mathbf{d}_1 \times \mathbf{d}_2| = \left| \begin{vmatrix} i & j & \mathbf{k} \\ 0 & 2 & -2 \\ 3 & 0 & 1 \end{vmatrix} \right| = \left| 2\mathbf{i} - 6\mathbf{j} - 6\mathbf{k} \right| = \sqrt{76} = 2\sqrt{19}
\]

2. Let the pressure $P$ and temperature $T$ at a point $(x, y, z)$ be

\[
P(x, y, z) = \frac{x^2 + 2y^2}{1 + z^2}, \quad T(x, y, z) = 5 + xy - z^2
\]

(a) If the position of an airplane at time $t$ is

\[
(x(t), y(t), z(t)) = (2t, t^2 - 1, \cos t)
\]

find $\frac{d}{dt}(PT)^2$ at time $t = 0$ as observed from the airplane.

(b) In which direction should a bird at the point $(0, -1, 1)$ fly if it wants to keep both $P$ and $T$ constant. (Give one possible direction vector. It does not need to be a unit vector.)

(c) An ant crawls on the surface $z^3 + zx + y^2 = 2$. When the ant is at the point $(0, -1, 1)$, in which direction should it go for maximum increase of the temperature $T = 5 + xy - z^2$? Your answer should be a vector $\langle a, b, c \rangle$, not necessarily of unit length. (Note that the ant cannot crawl in the direction of the gradient because that leads off the surface. The direction vector $\langle a, b, c \rangle$ has to be on the tangent plane to the surface.)

Solution. Reading through the question as a whole we see that we will need

- for part (a), the gradient of $PT$ at $(2t, t^2 - 1, \cos t)|_{t=0} = (0, -1, 1)$
- for part (b), the gradients of both $P$ and $T$ at $(0, -1, 1)$ and
• for part (c), the gradient of $T$ at $(0, -1, 1)$ and the gradient of $S = z^3 + xz + y^2$ at $(0, -1, 1)$ (to get the normal vector to the surface at that point).

So, by way of preparation, let’s compute all of these gradients.

$$\nabla P(x, y, z) = \frac{2x}{1 + z^2}i + \frac{4y}{1 + z^2}j - \frac{(x^2 + 2y^2)2z}{(1 + z^2)^2}k \quad \nabla P(0, -1, 1) = -2j - \hat{k}$$
$$\nabla T(x, y, z) = y\hat{i} + x\hat{j} - 2z\hat{k} \quad \nabla T(0, -1, 1) = -\hat{i} - 2\hat{k}$$
$$\nabla S(x, y, z) = z\hat{i} + 2y\hat{j} + (x + 3z^2)\hat{k} \quad \nabla S(0, -1, 1) = \hat{i} - 2\hat{j} + 3\hat{k}$$

To get the gradient of $PT$ we use the product rule

$$\nabla (PT)(x, y, z) = T(x, y, z)\nabla P(x, y, z) + P(x, y, z)\nabla T(x, y, z)$$

so that

$$\nabla (PT)(0, -1, 1) = T(0, -1, 1)\nabla P(0, -1, 1) + P(0, -1, 1)\nabla T(0, -1, 1)$$

$$= (5 + 0 - 1)(-2\hat{j} - \hat{k}) + \frac{0 + 2}{1 + 1}(-\hat{i} - 2\hat{k})$$

$$= -\hat{i} - 8\hat{j} - 6\hat{k}$$

(a) Since \(\frac{d}{dt}(PT)^2 = 2(PT)\frac{d}{dt}(PT)\), and the velocity vector of the plane at time 0 is

$$\frac{d}{dt} \left( 2t, t^2 - 1, \cos t \right) \bigg|_{t=0} = \left( 2, 2t, -\sin t \right) \bigg|_{t=0} = \langle 2, 0, 0 \rangle$$

we have

$$\frac{d}{dt} \left( PT \right)^2 \bigg|_{t=0} = 2P(0, -1, 1)T(0, -1, 1) \nabla (PT)(0, -1, 1) \cdot \langle 2, 0, 0 \rangle$$

$$= 2 \frac{0 + 2}{1 + 1} (5 + 0 - 1) \langle -1, -8, -6 \rangle \cdot \langle 2, 0, 0 \rangle$$

$$= -16$$

(b) The direction should be perpendicular to $\nabla P(0, -1, 1)$ (to keep $P$ constant) and should also be perpendicular to $\nabla T(0, -1, 1)$ (to keep $T$ constant). So any nonzero constant times

$$\pm \nabla P(0, -1, 1) \times \nabla T(0, -1, 1) = \pm \langle 0, -2, -1 \rangle \times \langle -1, 0, -2 \rangle = \pm \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$= \pm \langle 4, 1, -2 \rangle$$

are allowed directions.

(c) We want the direction to be as close as possible to $\nabla T(0, -1, 1) = \langle -1, 0, -2 \rangle$ while still being tangent to the surface, i.e. being perpendicular to the normal vector $\nabla S(0, -1, 1) = \langle 1, -2, 3 \rangle$. We can get that optimal direction by subtracting from $\nabla T(0, -1, 1)$ the projection of $\nabla T(0, -1, 1)$ onto the normal vector.
The projection of $\nabla T(0, -1, 1)$ onto the normal vector $\nabla S(0, -1, 1)$ is

$$\text{proj}_{\nabla S(0, -1, 1)} \nabla T(0, -1, 1) = \frac{\nabla T(0, -1, 1) \cdot \nabla S(0, -1, 1)}{|\nabla S(0, -1, 1)|^2} \nabla S(0, -1, 1)$$

$$= \frac{\langle -1, 0, -2 \rangle \cdot \langle 1, -2, 3 \rangle}{|\langle 1, -2, 3 \rangle|^2} \langle 1, -2, 3 \rangle$$

$$= -\frac{7}{14} \langle 1, -2, 3 \rangle$$

So the optimal direction is

$$d = \nabla T(0, -1, 1) - \text{proj}_{\nabla S(0, -1, 1)} \nabla T(0, -1, 1)$$

$$= \langle -1, 0, -2 \rangle - -\frac{7}{14} \langle 1, -2, 3 \rangle$$

$$= \langle -\frac{1}{2}, -1, -\frac{1}{2} \rangle$$

So any positive non zero multiple of $-\langle 1, 2, 1 \rangle$ will do. Note, as a check, that $-\langle 1, 2, 1 \rangle$ has dot product zero, i.e. is perpendicular to, $\nabla S(0, -1, 1) = \langle 1, -2, 3 \rangle$.

3. Consider the function

$$f(x, y) = 3kx^2y + y^3 - 3x^2 - 3y^2 + 4$$

where $k > 0$ is a constant. Find and classify all critical points of $f(x, y)$ as local minima, local maxima, saddle points or points of indeterminate type. Carefully distinguish the cases $k < \frac{1}{2}$, $k = \frac{1}{2}$ and $k > \frac{1}{2}$.

**Solution.** To find the critical points we will need the gradient of $f$ and to apply the second derivative test of Theorem 2.9.16 in the CLP–III text we will need all second order
partial derivatives. So we need all partial derivatives of \( f \) up to order two. Here they are.

\[
\begin{align*}
  f &= 3kx^2y + y^3 - 3x^2 - 3y^2 + 4 \\
  f_x &= 6kxy - 6x \\
  f_y &= 3kx^2 + 3y^2 - 6y \\
  f_{xx} &= 6ky - 6 \\
  f_{yy} &= 6y - 6 \\
  f_{xy} &= 6kx
\end{align*}
\]

(Of course, \( f_{xy} \) and \( f_{yx} \) have to be the same. It is still useful to compute both, as a way to catch some mechanical errors.)

The critical points are the solutions of

\[
\begin{align*}
  f_x &= 6x(ky - 1) = 0 \\
  f_y &= 3kx^2 + 3y^2 - 6y = 0
\end{align*}
\]

The first equation is satisfied if at least one of \( x = 0 \), \( y = 1/k \) are satisfied. (Recall that \( k > 0 \).)

- If \( x = 0 \), the second equation reduces to \( 3y(y - 2) = 0 \), which is satisfied if either \( y = 0 \) or \( y = 2 \).
- If \( y = 1/k \), the second equation reduces to \( 3kx^2 + \frac{3}{k^2} - \frac{6}{k} = 3kx^2 + \frac{3}{k^2}(1 - 2k) = 0 \).

**Case \( k < \frac{1}{2} \):** If \( k < \frac{1}{2} \), then \( \frac{3}{k^2}(1 - 2k) > 0 \) and the equation \( 3kx^2 + \frac{3}{k^2}(1 - 2k) = 0 \) has no real solutions. In this case there are two critical points: \((0,0), (0,2)\) and the classification is

<table>
<thead>
<tr>
<th>critical point</th>
<th>( f_{xx}f_{yy} - f_{xy}^2 )</th>
<th>( f_{xx} )</th>
<th>type</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0,0)</td>
<td>((-6) \times (-6) - (0)^2 &gt; 0)</td>
<td>(-6)</td>
<td>local max</td>
</tr>
<tr>
<td>(0,2)</td>
<td>((12k - 6) \times 6 - (0)^2 &lt; 0)</td>
<td></td>
<td>saddle point</td>
</tr>
</tbody>
</table>

**Case \( k = \frac{1}{2} \):** If \( k = \frac{1}{2} \), then \( \frac{3}{k^2}(1 - 2k) = 0 \) and the equation \( 3kx^2 + \frac{3}{k^2}(1 - 2k) = 0 \) reduces to \( 3kx^2 = 0 \) which has as its only solution \( x = 0 \). We have already seen this third critical point, \( x = 0, y = 1/k = 2 \). So there are again two critical points: \((0,0), (0,2)\) and the classification is

<table>
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<tr>
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<tr>
<td>(0,2)</td>
<td>((12k - 6) \times 6 - (0)^2 = 0)</td>
<td></td>
<td>unknown</td>
</tr>
</tbody>
</table>

**Case \( k > \frac{1}{2} \):** If \( k > \frac{1}{2} \), then \( \frac{3}{k^2}(1 - 2k) < 0 \) and the equation \( 3kx^2 + \frac{3}{k^2}(1 - 2k) = 0 \) reduces to \( 3kx^2 = \frac{3}{k^2}(2k - 1) \) which has two solutions, namely \( x = \pm \frac{1}{k} \sqrt{(2k - 1)} \). So there are four critical points: \((0,0), (0,2), \left(\sqrt{\frac{1}{k^2}(2k - 1)}, \frac{1}{k}\right) \) and \( \left(-\sqrt{\frac{1}{k^2}(2k - 1)}, \frac{1}{k}\right) \) and the classification is
4. Find the largest and smallest values of

\[ f(x, y, z) = 6x + y^2 + xz \]

on the sphere \( x^2 + y^2 + z^2 = 36 \). Determine all points at which these values occur.

**Solution 1.** This is a constrained optimization problem with objective function \( f(x, y, z) = 6x + y^2 + xz \) and constraint function \( g(x, y, z) = x^2 + y^2 + z^2 - 36 \). By Theorem 2.10.2 in the CLP–III text, any local minimum or maximum \((x, y, z)\) must obey the Lagrange multiplier equations

\[
\begin{align*}
    f_x &= 6 + z = 2\lambda x = \lambda g_x \\
    f_y &= 2y = 2\lambda y = \lambda g_y \\
    f_z &= x = 2\lambda z = \lambda g_z \\
    x^2 + y^2 + z^2 &= 36
\end{align*}
\]

for some real number \( \lambda \). By equation (E2), \( y(1 - \lambda) = 0 \), which is obeyed if and only if at least one of \( y = 0 \), \( \lambda = 1 \) is obeyed.

- If \( y = 0 \), the remaining equations reduce to

\[
\begin{align*}
    6 + z &= 2\lambda x \\
    x &= 2\lambda z \\
    x^2 + z^2 &= 36
\end{align*}
\]

Substituting (E3) into (E1) gives \( 6 + z = 4\lambda^2 z \), which forces \( 4\lambda^2 \neq 1 \) (since \( 6 \neq 0 \)) and gives \( z = \frac{6}{4\lambda^2 - 1} \) and then \( x = \frac{12\lambda}{4\lambda^2 - 1} \). Substituting this into (E4) gives

\[
\frac{144\lambda^2}{(4\lambda^2 - 1)^2} + \frac{36}{(4\lambda^2 - 1)^2} = 36
\]

\[
\frac{4\lambda^2}{(4\lambda^2 - 1)^2} + \frac{1}{(4\lambda^2 - 1)^2} = 1
\]

\[
4\lambda^2 + 1 = (4\lambda^2 - 1)^2
\]
Write $\mu = 4\lambda^2$. Then this last equation is

$$\mu + 1 = \mu^2 - 2\mu + 1 \iff \mu^2 - 3\mu = 0 \iff \mu = 0, 3$$

When $\mu = 0$, we have $z = \frac{6}{\mu - 1} = -6$ and $x = 0$ (by (E4)). When $\mu = 3$, we have $z = \frac{6}{\mu - 1} = 3$ and then $x = \pm\sqrt{27} = \pm3\sqrt{3}$ (by (E4)).

- If $\lambda = 1$, the remaining equations reduce to
  
  $$6 + z = 2x \quad (E1)$$
  $$x = 2z \quad (E3)$$
  $$x^2 + y^2 + z^2 = 36 \quad (E4)$$

  Substituting (E3) into (E1) gives $6 + z = 4z$ and hence $z = 2$. Then (E3) gives $x = 4$ and (E4) gives $4^2 + y^2 + 2^2 = 36$ or $y^2 = 16$ or $y = \pm 4$.

  So we have the following candidates for the locations of the min and max

<table>
<thead>
<tr>
<th>point</th>
<th>(0, 0, -6)</th>
<th>(3\sqrt{3}, 0, 3)</th>
<th>(-3\sqrt{3}, 0, 3)</th>
<th>(4, 4, 2)</th>
<th>(4, -4, 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>value of $f$</td>
<td>0</td>
<td>27\sqrt{3}</td>
<td>-27\sqrt{3}</td>
<td>48</td>
<td>48</td>
</tr>
<tr>
<td></td>
<td>min</td>
<td>max</td>
<td>max</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Solution 2.** On the sphere we have $y^2 = 36 - x^2 - z^2$ and hence $f = 36 + 6x + xz - x^2 - z^2$ and $x^2 + z^2 \leq 36$. So it suffices to find the max and min of $h(x, z) = 36 + 6x + xz - x^2 - z^2$ on the disk $D = \{(x, z) \mid x^2 + z^2 \leq 36\}$.

- If a max or min occurs at an interior point $(x, z)$ of $D$, then $(x, z)$ must be a critical point of $h$ and hence must obey
  
  $$h_x = 6 + z - 2x = 0$$
  $$h_z = x - 2z = 0$$

  Substituting $x = 2z$ into the first equation gives $6 - 3z = 0$ and hence $z = 2$ and $x = 4$.

- If a max or min occurs at a point $(x, z)$ on the boundary of $D$, we have $x^2 + z^2 = 36$ and hence $x = \pm\sqrt{36 - z^2}$ and $h = 6x + xz = \pm(6 + z)\sqrt{36 - z^2}$ with $-6 \leq z \leq 6$. So the max or min can occur either when $z = -6$ or $z = +6$ or at a $z$ obeying

$$0 = \frac{d}{dz}[(6 + z)\sqrt{36 - z^2}] = \sqrt{36 - z^2} - \frac{z(6 + z)}{\sqrt{36 - z^2}}$$
or equivalently

\[ 36 - z^2 - z(6 + z) = 0 \]
\[ 2z^2 + 6z - 36 = 0 \]
\[ z^2 + 3z - 18 = 0 \]
\[ (z + 6)(z - 3) = 0 \]

So the max or min can occur either when \( z = \pm 6 \) or \( z = 3 \).

So we have the following candidates for the locations of the min and max

<table>
<thead>
<tr>
<th>point</th>
<th>(0,0,±6)</th>
<th>(3(\sqrt{3}),0,3)</th>
<th>(−3(\sqrt{3}),0,3)</th>
<th>(4,4,2)</th>
<th>(4,−4,2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>value of f</td>
<td>0</td>
<td>27(\sqrt{3})</td>
<td>−27(\sqrt{3})</td>
<td>48</td>
<td>48</td>
</tr>
<tr>
<td></td>
<td>min</td>
<td>max</td>
<td>max</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. Let \( D \) be the region in the \( xy \)–plane bounded on the left by the line \( x = 2 \) and on the right by the circle \( x^2 + y^2 = 16 \). Evaluate

\[
\iint_D (x^2 + y^2)^{-3/2} \, dA
\]

**Solution.** Here is a sketch of \( D \).

We’ll use polar coordinates. In polar coordinates the circle \( x^2 + y^2 = 16 \) is \( r = 4 \) and the line \( x = 2 \) is \( r \cos \theta = 2 \). So

\[
D = \left\{ (r \cos \theta , r \sin \theta) \mid -\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3} , \quad \frac{2}{\cos \theta} \leq r \leq 4 \right\}
\]
and, as $dA = r \, dr \, d\theta$, the specified integral is

\[
\int\int_D (x^2 + y^2)^{-3/2} \, dA = \int_{-\pi/3}^{\pi/3} d\theta \int_{2/\cos\theta}^4 dr \, \frac{r^1}{r^3}
\]

\[
= \int_{-\pi/3}^{\pi/3} d\theta \left[ -\frac{1}{r} \right]_{2/\cos\theta}^4
\]

\[
= \int_{-\pi/3}^{\pi/3} d\theta \left[ \frac{\cos \theta}{2} - \frac{1}{4} \right]
\]

\[
= \left[ \frac{\sin \theta}{2} - \frac{\theta^{\pi/3}}{4} \right]_{-\pi/3}
\]

\[
= \frac{\sqrt{3}}{2} - \frac{\pi}{6}
\]

6. (a) Let

\[ I = \int_0^2 \int_0^x f(x, y) \, dy \, dx + \int_2^6 \int_0^{\sqrt{6-x}} f(x, y) \, dy \, dx \]

Express $I$ as an integral where we integrate first with respect to $x$.

(b) Let

\[ J = \int_0^1 \int_0^y \int_0^{\sqrt{6-x}} f(x, y, z) \, dz \, dy \, dx \]

Express $J$ as an integral where the integrations are to be performed in the order $x$ first, then $y$, then $z$.

Solution. (a) We first have to get a picture of the domain of integration. The first integral has domain of integration

\[ \{ (x, y) \mid 0 \leq x \leq 2, \ 0 \leq y \leq x \} \]

and the second integral has domain of integration

\[ \{ (x, y) \mid 2 \leq x \leq 6, \ 0 \leq y \leq \sqrt{6-x} \} \]

Here is a sketch. The domain of integration for the first integral is the shaded triangular region to the left of $x = 2$ and the domain of integration for the second integral is the shaded region to the right of $x = 2$. 
To exchange the order of integration, we use horizontal slices as in the figure below.

The bottom slice has $y = 0$ and the top slice has $y = 2$. On the slice at height $y$, $x$ runs from $y$ to $6 - y^2$. So

$$I = \int_0^2 \int_y^{6-y^2} f(x, y) \, dx \, dy$$

(b) In the integral $J$,

- $x$ runs from 0 to 1. In inequalities, $0 \leq x \leq 1$.
- Then, for each fixed $x$ in that range, $y$ runs from 0 to $x$. In inequalities, $0 \leq y \leq x$.
- Then, for each fixed $x$ and $y$ in those ranges, $z$ runs from 0 to $y$. In inequalities, $0 \leq z \leq y$.

These inequalties can be combined into

$$0 \leq z \leq y \leq x \leq 1 \quad (*)$$

We wish to reverse the order of integration so that the $z$–integral is on the outside, the $y$–integral is in the middle and the $x$–integral is on the inside.
• The smallest $z$ compatible with $(\ast)$ is $z = 0$ and the largest $z$ compatible with $(\ast)$ is $z = 1$ (when $x = y = z = 1$). So $0 \leq z \leq 1$.

• Then, for each fixed $z$ in that range, $(x, y)$ run over $z \leq y \leq x \leq 1$. In particular, the smallest allowed $y$ is $y = z$ and the largest allowed $y$ is $y = 1$ (when $x = y = 1$). So $z \leq y \leq 1$.

• Then, for each fixed $y$ and $z$ in those ranges, $x$ runs over $y \leq x \leq 1$.

\[ J = \int_0^1 \int_z^1 \int_y^1 f(x, y, z) \, dx \, dy \, dz \]

7. Let $E$ be the solid lying above the surface $z = y^2$ and below the surface $z = 4 - x^2$.

Evaluate \[ \iiint_E y^2 \, dV \]

*Hint:* you may need to use the half angle formulas:

\[
\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}, \quad \cos^2 \theta = \frac{1 + \cos(2\theta)}{2}
\]

**Solution.** Note that the surfaces meet when $z = y^2 = 4 - x^2$ and then $(x, y)$ runs over the circle $x^2 + y^2 = 4$. So the domain of integration is

\[ E = \{ (x, y, z) \mid x^2 + y^2 \leq 4, \ y^2 \leq z \leq 4 - x^2 \} \]

Let’s switch to cylindrical coordinates. Then

\[ E = \{ (r \cos \theta, r \sin \theta, z) \mid 0 \leq r \leq 2, \ 0 \leq \theta \leq 2\pi, \ r^2 \sin^2 \theta \leq z \leq 4 - r^2 \cos^2 \theta \} \]

and, since $dV = r \, dr \, d\theta \, dz$,

\[
\begin{align*}
\iiint_E y^2 \, dV &= \int_0^2 dr \int_0^{2\pi} d\theta \int_{r^2 \sin^2 \theta}^{4 - r^2 \cos^2 \theta} dz \ r \ r^2 \sin^2 \theta \\
&= \int_0^2 dr \int_0^{2\pi} d\theta \ r^3 \sin^2 \theta \left[ 4 - r^2 \cos^2 \theta - r^2 \sin^2 \theta \right] \\
&= \int_0^2 dr \ [4r^3 - r^5] \int_0^{2\pi} d\theta \ \frac{1 - \cos(2\theta)}{2} \\
&= \frac{1}{2} \int_0^2 dr \ [4r^3 - r^5] \left[ \theta - \frac{\sin(2\theta)}{2} \right]_0^{2\pi} \\
&= \pi \left[ r^4 - \frac{r^6}{6} \right]_0^2 \\
&= \frac{16\pi}{3}
\end{align*}
\]
For an efficient, sneaky, way to evaluate $\int_0^{2\pi} \sin^2 \theta \, d\theta$, see Remark 3.3.5 in the CLP–III text.

8. Let $E$ be the solid

$$0 \leq z \leq \sqrt{x^2 + y^2}, \quad x^2 + y^2 \leq 1,$$

and consider the integral

$$I = \iiint_E z \sqrt{x^2 + y^2 + z^2} \, dV.$$

(a) Write the integral $I$ in cylindrical coordinates.

(b) Write the integral $I$ in spherical coordinates.

(c) Evaluate the integral $I$ using either form.

**Solution.** (a) In cylindrical coordinates $0 \leq z \leq \sqrt{x^2 + y^2}$ becomes $0 \leq z \leq r$, and $x^2 + y^2 \leq 1$ becomes $0 \leq r \leq 1$. So

$$E = \{ (r \cos \theta, r \sin \theta, z) \mid 0 \leq r \leq 1, \ 0 \leq \theta \leq 2\pi, \ 0 \leq z \leq r \}$$

and, since $dV = r \, dr \, d\theta \, dz$,

$$I = \iiint_E z \sqrt{x^2 + y^2 + z^2} \, dV = \int_0^1 \int_0^{2\pi} \int_0^r z \sqrt{r^2 + z^2} \, dr \, d\theta \, dz.$$

(b) Here is a sketch of a constant $\theta$ section of $E$.

Recall that the spherical coordinate $\varphi$ is the angle between the $z$–axis and the radius vector. So, in spherical coordinates $z = r$ (which makes an angle $\frac{\pi}{4}$ with the $z$ axis) becomes $\varphi = \frac{\pi}{4}$, and the plane $z = 0$, i.e. the $xy$–plane, becomes $\varphi = \frac{\pi}{2}$, and $r = 1$ becomes $\rho \sin \varphi = 1$. So

$$E = \left\{ (\rho \sin \varphi \cos \theta, \rho \sin \varphi \sin \theta, \rho \cos \varphi) \left| \frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}, \ 0 \leq \theta \leq 2\pi, \ 0 \leq \rho \leq \frac{1}{\sin \varphi} \right. \right\}$$
and, since $dV = \rho^2 \sin \varphi \, d\rho \, d\theta \, d\varphi$,

$$I = \iiint_E z \sqrt{x^2 + y^2 + z^2} \, dV = \int_{\pi/4}^{\pi/2} d\varphi \int_0^{2\pi} d\theta \int_0^{1/\sin \varphi} d\rho \, \rho^2 \sin \varphi \, \rho \cos \varphi \, \rho$$

$$= \int_{\pi/4}^{\pi/2} d\varphi \int_0^{2\pi} d\theta \int_0^{1/\sin \varphi} d\rho \, \rho^4 \sin \varphi \cos \varphi$$

(c) We’ll integrate using the spherical coordinate version.

$$I = \int_{\pi/4}^{\pi/2} d\varphi \int_0^{2\pi} d\theta \int_0^{1/\sin \varphi} d\rho \, \rho^4 \sin \varphi \cos \varphi$$

$$= \int_{\pi/4}^{\pi/2} d\varphi \int_0^{2\pi} d\theta \frac{1}{5 \sin^5 \varphi} \sin \varphi \cos \varphi$$

$$= \frac{2\pi}{5} \int_{\pi/4}^{\pi/2} d\varphi \frac{\cos \varphi}{\sin^4 \varphi}$$

$$= \frac{2\pi}{5} \int_{1/\sqrt{2}}^1 \frac{du}{u^4} \quad \text{with} \ u = \sin \varphi, \ du = \cos \varphi \, d\varphi$$

$$= \frac{2\pi}{5} \left[ \frac{u^{-3}}{-3} \right]_{1/\sqrt{2}}^1$$

$$= \frac{2(2\sqrt{2} - 1)\pi}{15}$$