MATHEMATICS 200 April 2014 Final Exam Solutions

1. Consider two planes W_1, W_2, and a line M defined by:

$$W_1 \ : \ -2x + y + z = 7, \qquad W_2 \ : \ -x + 3y + 3z = 6, \qquad M \ : \ \frac{x}{2} = \frac{2y-4}{4} = z + 5.$$

(a) Find a parametric equation of the line of intersection L of W_1 and W_2.

(b) Find the distance from L to M .

(c) Find the area of the parallelogram on W_2 $(-x + 3y + 3z = 6)$ defined by $0 \le x \le 3$, $0 \le y \le 2$.

Solution. (a) Let's use z as the parameter and rename it to t. That is, $z = t$. Subtracting 2 times the W_2 equation from the W_1 equation gives

$$-5y - 5z = -5 \implies y = 1 - z = 1 - t$$

Substituting the result into the equation for W_2 gives

$$-x + 3(1-t) + 3t = 6 \implies x = -3$$

So a parametric equation is

$$(x, y, z) = (-3, 1, 0) + t\,\langle 0, -1, 1\rangle$$

(b) *Solution 1*

We can also parametrize M using $z = t$:

$$x = 2z + 10 = 2t + 10, \ y = 2z + 12 = 2t + 12 \qquad \implies \qquad (x, y, z) = (10, 12, 0) + t\,\langle 2, 2, 1\rangle$$

So one point on M is $(10, 12, 0)$ and one point on L is $(-3, 1, 0)$ and

$$\mathbf{v} = \langle (-3) - (10)\,,\ (1) - (12)\,,\ 0 - 0\rangle = \langle -13\,,\ -11\,,\ 0\rangle$$

is one vector from a point on M to a point on L.

The direction vectors of L and M are $\langle 0, -1, 1\rangle$ and $\langle 2, 2, 1\rangle$, respectively. The vector

$$\mathbf{n} = \langle 0, -1, 1\rangle \times \langle 2, 2, 1\rangle = \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 0 & -1 & 1 \\ 2 & 2 & 1 \end{bmatrix} = \langle -3\,,\ 2\,,\ 2\rangle$$

is then perpendicular to both L and M.

The distance from L to M is then the length of the projection of $\mathbf{v}$ on $\mathbf{n}$, which is

$$\frac{|\mathbf{v}\cdot\mathbf{n}|}{|\mathbf{n}|} = \frac{|39 - 22 + 0|}{\sqrt{9+4+4}} = \sqrt{17}$$

(b) *Solution 2* We can also parametrize M using $z = s$:

$$x = 2z + 10 = 2s + 10,\ y = 2z + 12 = 2s + 12 \quad \Longrightarrow \quad (x, y, z) = (10, 12, 0) + s\,\langle 2, 2, 1\rangle$$

The vector from the point $(-3, 1, 0) + t\,\langle 0, -1, 1\rangle$ on L to the point $(10, 12, 0) + s\,\langle 2, 2, 1\rangle$ on M is

$$\langle 13 + 2s\,,\ 11 + 2s + t\,,\ s - t\rangle$$

So the distance from the point $(-3, 1, 0) + t\,\langle 0, -1, 1\rangle$ on L to the point $(10, 12, 0) + s\,\langle 2, 2, 1\rangle$ on M is the square root of

$$D(s, t) = (13 + 2s)^2 + (11 + 2s + t)^2 + (s - t)^2$$

That distance is minimized when

$$\begin{aligned} 0 &= \frac{\partial D}{\partial s} = 4(13 + 2s) + 4(11 + 2s + t) + 2(s - t) \\ 0 &= \frac{\partial D}{\partial t} = 2(11 + 2s + t) - 2(s - t) \end{aligned}$$

Cleaning up those equations gives

$$\begin{aligned} 18s + 2t &= -96 \\ 2s + 4t &= -22 \end{aligned}$$

or

$$9s + t = -48 \tag{E1}$$
$$s + 2t = -11 \tag{E2}$$

Subtracting (E2) from twice (E1) gives

$$17s = -85 \implies s = -5$$

Substituting that into (E2) gives

$$2t = -11 + 5 \implies t = -3$$

Note that

$$\begin{aligned} 13 + 2s &= 3 \\ 11 + 2s + t &= -2 \\ s - t &= -2 \end{aligned}$$

So the distance is

$$\sqrt{D(-5, -3)} = \sqrt{3^2 + (-2)^2 + (-2)^2} = \sqrt{17}$$

(c) Note that

- the point on W_2 with $x = 0$, $y = 0$ obeys $-0 + 3(0) + 3z = 6$ and so has $z = 2$
- the point on W_2 with $x = 0$, $y = 2$ obeys $-0 + 3(2) + 3z = 6$ and so has $z = 0$
- the point on W_2 with $x = 3$, $y = 0$ obeys $-3 + 3(0) + 3z = 6$ and so has $z = 3$
- the point on W_2 with $x = 3$, $y = 2$ obeys $-3 + 3(2) + 3z = 6$ and so has $z = 1$

So the four corners of the parallelogram are $(0, 0, 2)$, $(0, 2, 0)$, $(3, 0, 3)$ and $(3, 2, 1)$. The vectors

$$\begin{aligned} \mathbf{d}_1 &= \langle 0-0\,,\, 2-0\,,\, 0-2\rangle = \langle 0\,,\, 2\,,\, -2\rangle \\ \mathbf{d}_2 &= \langle 3-0\,,\, 0-0\,,\, 3-2\rangle = \langle 3\,,\, 0\,,\, 1\rangle \end{aligned}$$

form two sides of the paralleogram. So the area of the parallelogram is

$$\left|\mathbf{d}_1 \times \mathbf{d}_2\right| = \left|\det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 0 & 2 & -2 \\ 3 & 0 & 1 \end{bmatrix}\right| = \left|2\hat{\boldsymbol{\imath}} - 6\hat{\boldsymbol{\jmath}} - 6\hat{\mathbf{k}}\right| = \sqrt{76} = 2\sqrt{19}$$

2. Let the pressure P and temperature T at a point (x, y, z) be

$$P(x, y, z) = \frac{x^2 + 2y^2}{1 + z^2}, \qquad T(x, y, z) = 5 + xy - z^2$$

(a) If the position of an airplane at time t is

$$(x(t), y(t), z(t)) = (2t, t^2 - 1, \cos t)$$

find $\frac{\mathrm{d}}{\mathrm{d}t}(PT)^2$ at time $t = 0$ as observed from the airplane.

(b) In which direction should a bird at the point $(0, -1, 1)$ fly if it wants to keep both P and T constant. (Give one possible direction vector. It does not need to be a unit vector.)

(c) An ant crawls on the surface $z^3 + zx + y^2 = 2$. When the ant is at the point $(0, -1, 1)$, in which direction should it go for maximum increase of the temperature $T = 5 + xy - z^2$? Your answer should be a vector $\langle a, b, c\rangle$, not necessarily of unit length. (Note that the ant cannot crawl in the direction of the gradient because that leads off the surface. The direction vector $\langle a, b, c\rangle$ has to be on the tangent plane to the surface.)

Solution. Reading through the question as a whole we see that we will need

- for part (a), the gradient of PT at $(2t, t^2 - 1, \cos t)\Big|_{t=0} = (0, -1, 1)$
- for part (b), the gradients of both P and T at $(0, -1, 1)$ and

- for part (c), the gradient of T at $(0,-1,1)$ and the gradient of $S = z^3 + xz + y^2$ at $(0,-1,1)$ (to get the normal vector to the surface at that point).

So, by way of preparation, let's compute all of these gradients.

$$\begin{aligned}
\boldsymbol{\nabla} P(x,y,z) &= \frac{2x}{1+z^2}\hat{\boldsymbol{\imath}} + \frac{4y}{1+z^2}\hat{\boldsymbol{\jmath}} - \frac{(x^2+2y^2)2z}{(1+z^2)^2}\hat{\mathbf{k}} & \boldsymbol{\nabla} P(0,-1,1) &= -2\,\hat{\boldsymbol{\jmath}} - \hat{\mathbf{k}} \\
\boldsymbol{\nabla} T(x,y,z) &= y\,\hat{\boldsymbol{\imath}} + x\,\hat{\boldsymbol{\jmath}} - 2z\,\hat{\mathbf{k}} & \boldsymbol{\nabla} T(0,-1,1) &= -\hat{\boldsymbol{\imath}} - 2\hat{\mathbf{k}} \\
\boldsymbol{\nabla} S(x,y,z) &= z\,\hat{\boldsymbol{\imath}} + 2y\,\hat{\boldsymbol{\jmath}} + (x+3z^2)\,\hat{\mathbf{k}} & \boldsymbol{\nabla} S(0,-1,1) &= \hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} + 3\,\hat{\mathbf{k}}
\end{aligned}$$

To get the gradient of PT we use the product rule

$$\boldsymbol{\nabla}(PT)(x,y,z) = T(x,y,z)\,\boldsymbol{\nabla} P(x,y,z) + P(x,y,z)\,\boldsymbol{\nabla} T(x,y,z)$$

so that

$$\begin{aligned}
\boldsymbol{\nabla}(PT)(0,-1,1) &= T(0,-1,1)\,\boldsymbol{\nabla} P(0,-1,1) + P(0,-1,1)\,\boldsymbol{\nabla} T(0,-1,1) \\
&= (5+0-1)\big(-2\,\hat{\boldsymbol{\jmath}} - \hat{\mathbf{k}}\big) + \frac{0+2}{1+1}\big(-\hat{\boldsymbol{\imath}} - 2\hat{\mathbf{k}}\big) \\
&= -\hat{\boldsymbol{\imath}} - 8\,\hat{\boldsymbol{\jmath}} - 6\,\hat{\mathbf{k}}
\end{aligned}$$

(a) Since $\frac{\mathrm{d}}{\mathrm{d}t}(PT)^2 = 2(PT)\frac{\mathrm{d}}{\mathrm{d}t}(PT)$, and the velocity vector of the plane at time 0 is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle 2t, t^2-1, \cos t\right\rangle\Big|_{t=0} = \langle 2, 2t, -\sin t\rangle\Big|_{t=0} = \langle 2,0,0\rangle$$

we have

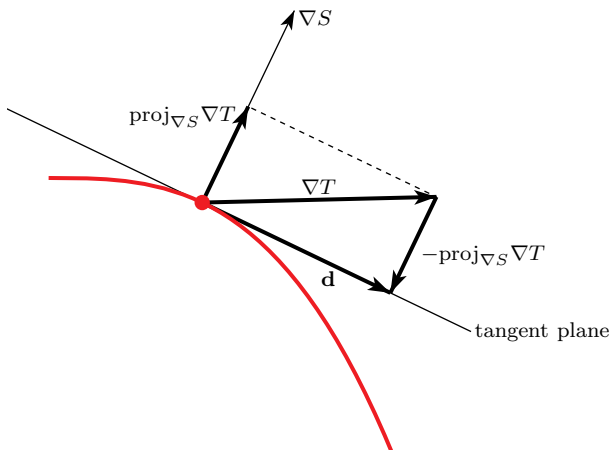
$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(PT)^2\Big|_{t=0} &= 2\,P(0,-1,1)\,T(0,-1,1)\;\boldsymbol{\nabla}(PT)(0,-1,1)\cdot\langle 2,0,0\rangle \\
&= 2\;\frac{0+2}{1+1}\;(5+0-1)\,\langle -1,-8,-6\rangle\cdot\langle 2,0,0\rangle \\
&= -16
\end{aligned}$$

(b) The direction should be perpendicular to $\boldsymbol{\nabla} P(0,-1,1)$ (to keep P constant) and should also be perpendicular to $\boldsymbol{\nabla} T(0,-1,1)$ (to keep T constant). So any nonzero constant times

$$\begin{aligned}
\pm\boldsymbol{\nabla} P(0,-1,1)\times\boldsymbol{\nabla} T(0,-1,1) &= \pm\langle 0\,,\,-2\,,\,-1\rangle\times\langle -1\,,\,0\,,\,-2\rangle = \pm\det\begin{bmatrix}\hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 0 & 2 & 1 \\ 1 & 0 & 2\end{bmatrix} \\
&= \pm\langle 4\,,\,1\,,\,-2\rangle
\end{aligned}$$

are allowed directions.

(c) We want the direction to be as close as possible to $\boldsymbol{\nabla} T(0,-1,1) = \langle -1\,,\,0\,,\,-2\rangle$ while still being tangent to the surface, i.e. being perpendicular to the normal vector $\boldsymbol{\nabla} S(0,-1,1) = \langle 1\,,\,-2\,,\,3\rangle$. We can get that optimal direction by subtracting from $\boldsymbol{\nabla} T(0,-1,1)$ the projection of $\boldsymbol{\nabla} T(0,-1,1)$ onto the normal vector.



The projection of $\boldsymbol{\nabla} T(0,-1,1)$ onto the normal vector $\boldsymbol{\nabla} S(0,-1,1)$ is

$$\begin{aligned}
\text{proj}_{\boldsymbol{\nabla} S(0,-1,1)}\boldsymbol{\nabla} T(0,-1,1) &= \frac{\boldsymbol{\nabla} T(0,-1,1)\cdot\boldsymbol{\nabla} S(0,-1,1)}{|\boldsymbol{\nabla} S(0,-1,1)|^2}\boldsymbol{\nabla} S(0,-1,1) \\
&= \frac{\langle -1\,,\,0\,,\,-2\rangle\cdot\langle 1\,,\,-2\,,\,3\rangle}{|\,\langle 1\,,\,-2\,,\,3\rangle\,|^2}\,\langle 1\,,\,-2\,,\,3\rangle \\
&= \frac{-7}{14}\,\langle 1\,,\,-2\,,\,3\rangle
\end{aligned}$$

So the optimal direction is

$$\begin{aligned}
\mathbf{d} &= \boldsymbol{\nabla} T(0,-1,1) - \text{proj}_{\boldsymbol{\nabla} S(0,-1,1)}\boldsymbol{\nabla} T(0,-1,1) \\
&= \langle -1\,,\,0\,,\,-2\rangle - \frac{-7}{14}\,\langle 1\,,\,-2\,,\,3\rangle \\
&= \left\langle -\frac{1}{2}\,,\,-1\,,\,-\frac{1}{2}\right\rangle
\end{aligned}$$

So any positive non zero multiple of $-\langle 1\,,\,2\,,\,1\rangle$ will do. Note, as a check, that $-\langle 1\,,\,2\,,\,1\rangle$ has dot product zero, i.e. is perpendicular to, $\boldsymbol{\nabla} S(0,-1,1) = \langle 1\,,\,-2\,,\,3\rangle$.

3. Consider the function

$$f(x,y) = 3kx^2y + y^3 - 3x^2 - 3y^2 + 4$$

where $k > 0$ is a constant. Find and classify all critical points of $f(x,y)$ as local minima, local maxima, saddle points or points of indeterminate type. Carefully distinguish the cases $k < \frac{1}{2}$, $k = \frac{1}{2}$ and $k > \frac{1}{2}$.

Solution. To find the critical points we will need the gradient of f and to apply the second derivative test of Theorem 2.9.16 in the CLP–III text we will need all second order

partial derivatives. So we need all partial derivatives of f up to order two. Here they are.

$$\begin{aligned} f &= 3kx^2y + y^3 - 3x^2 - 3y^2 + 4 \\ f_x &= 6kxy - 6x & f_{xx} &= 6ky - 6 & f_{xy} &= 6kx \\ f_y &= 3kx^2 + 3y^2 - 6y & f_{yy} &= 6y - 6 & f_{yx} &= 6kx \end{aligned}$$

(Of course, f_{xy} and f_{yx} have to be the same. It is still useful to compute both, as a way to catch some mechanical errors.)

The critical points are the solutions of

$$f_x = 6x(ky - 1) = 0 \qquad f_y = 3kx^2 + 3y^2 - 6y = 0$$

The first equation is satisfied if at least one of $x = 0$, $y = 1/k$ are satisfied. (Recall that $k > 0$.)

- If $x = 0$, the second equation reduces to $3y(y - 2) = 0$, which is satisfied if either $y = 0$ or $y = 2$.
- If $y = 1/k$, the second equation reduces to $3kx^2 + \frac{3}{k^2} - \frac{6}{k} = 3kx^2 + \frac{3}{k^2}(1 - 2k) = 0$.

Case $k < \frac{1}{2}$: If $k < \frac{1}{2}$, then $\frac{3}{k^2}(1-2k) > 0$ and the equation $3kx^2 + \frac{3}{k^2}(1-2k) = 0$ has no real solutions. In this case there are two critical points: $(0, 0)$, $(0, 2)$ and the classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0, 0)$	$(-6) \times (-6) - (0)^2 > 0$	-6	local max
$(0, 2)$	$(12k - 6) \times 6 - (0)^2 < 0$		saddle point

Case $k = \frac{1}{2}$: If $k = \frac{1}{2}$, then $\frac{3}{k^2}(1 - 2k) = 0$ and the equation $3kx^2 + \frac{3}{k^2}(1 - 2k) = 0$ reduces to $3kx^2 = 0$ which has as its only solution $x = 0$. We have already seen this third critical point, $x = 0$, $y = 1/k = 2$. So there are again two critical points: $(0, 0)$, $(0, 2)$ and the classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0, 0)$	$(-6) \times (-6) - (0)^2 > 0$	-6	local max
$(0, 2)$	$(12k - 6) \times 6 - (0)^2 = 0$		unknown

Case $k > \frac{1}{2}$: If $k > \frac{1}{2}$, then $\frac{3}{k^2}(1 - 2k) < 0$ and the equation $3kx^2 + \frac{3}{k^2}(1 - 2k) = 0$ reduces to $3kx^2 = \frac{3}{k^2}(2k - 1)$ which has two solutions, namely $x = \pm\sqrt{\frac{1}{k^3}(2k - 1)}$. So there are four critical points: $(0, 0)$, $(0, 2)$, $\left(\sqrt{\frac{1}{k^3}(2k - 1)}\,,\, \frac{1}{k}\right)$ and $\left(-\sqrt{\frac{1}{k^3}(2k - 1)}\,,\, \frac{1}{k}\right)$ and the classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,0)$	$(-6)\times(-6)-(0)^2>0$	-6	local max
$(0,2)$	$(12k-6)\times 6-(0)^2>0$	$12k-6>0$	local min
$\left(\sqrt{\frac{1}{k^3}(2k-1)}\,,\,\frac{1}{k}\right)$	$(6-6)\times(\frac{6}{k}-6)-(>0)^2<0$		saddle point
$\left(-\sqrt{\frac{1}{k^3}(2k-1)}\,,\,\frac{1}{k}\right)$	$(6-6)\times(\frac{6}{k}-6)-(<0)^2<0$		saddle point

4. Find the largest and smallest values of

$$f(x,y,z) = 6x + y^2 + xz$$

on the sphere $x^2+y^2+z^2=36$. Determine all points at which these values occur.

Solution 1. This is a constrained optimization problem with objective function $f(x,y,z) = 6x+y^2+xz$ and constraint function $g(x,y,z) = x^2+y^2+z^2-36$. By Theorem 2.10.2 in the CLP–III text, any local minimum or maximum (x,y,z) must obey the Lagrange multiplier equations

$$\begin{aligned} f_x = 6+z = 2\lambda x &= \lambda g_x && \text{(E1)}\\ f_y = 2y = 2\lambda y &= \lambda g_y && \text{(E2)}\\ f_z = x = 2\lambda z &= \lambda g_z && \text{(E3)}\\ x^2+y^2+z^2 &= 36 && \text{(E4)} \end{aligned}$$

for some real number λ. By equation (E2), $y(1-\lambda)=0$, which is obeyed if and only if at least one of $y=0$, $\lambda=1$ is obeyed.

- If $y=0$, the remaining equations reduce to

$$\begin{aligned} 6+z &= 2\lambda x && \text{(E1)}\\ x &= 2\lambda z && \text{(E3)}\\ x^2+z^2 &= 36 && \text{(E4)} \end{aligned}$$

Substituting (E3) into (E1) gives $6+z=4\lambda^2 z$, which forces $4\lambda^2\neq 1$ (since $6\neq 0$) and gives $z=\frac{6}{4\lambda^2-1}$ and then $x=\frac{12\lambda}{4\lambda^2-1}$. Substituting this into (E4) gives

$$\begin{aligned} \frac{144\lambda^2}{(4\lambda^2-1)^2}+\frac{36}{(4\lambda^2-1)^2} &= 36\\ \frac{4\lambda^2}{(4\lambda^2-1)^2}+\frac{1}{(4\lambda^2-1)^2} &= 1\\ 4\lambda^2+1 &= (4\lambda^2-1)^2 \end{aligned}$$

Write $\mu = 4\lambda^2$. Then this last equation is

$$\begin{aligned} \mu + 1 = \mu^2 - 2\mu + 1 &\iff \mu^2 - 3\mu = 0 \\ &\iff \mu = 0, 3 \end{aligned}$$

When $\mu = 0$, we have $z = \frac{6}{\mu - 1} = -6$ and $x = 0$ (by (E4)). When $\mu = 3$, we have $z = \frac{6}{\mu-1} = 3$ and then $x = \pm\sqrt{27} = \pm 3\sqrt{3}$ (by (E4)).

- If $\lambda = 1$, the remaining equations reduce to

$$6 + z = 2x \tag{E1}$$
$$x = 2z \tag{E3}$$
$$x^2 + y^2 + z^2 = 36 \tag{E4}$$

Substituting (E3) into (E1) gives $6 + z = 4z$ and hence $z = 2$. Then (E3) gives $x = 4$ and (E4) gives $4^2 + y^2 + 2^2 = 36$ or $y^2 = 16$ or $y = \pm 4$.

So we have the following candidates for the locations of the min and max

point	$(0, 0, -6)$	$(3\sqrt{3}, 0, 3)$	$(-3\sqrt{3}, 0, 3)$	$(4, 4, 2)$	$(4, -4, 2)$
value of f	0	$27\sqrt{3}$	$-27\sqrt{3}$	48	48
			min	max	max

Solution 2. On the sphere we have $y^2 = 36 - x^2 - z^2$ and hence $f = 36 + 6x + xz - x^2 - z^2$ and $x^2 + z^2 \leq 36$. So it suffices to find the max and min of $h(x, z) = 36 + 6x + xz - x^2 - z^2$ on the disk $D = \{ (x, z) \mid x^2 + z^2 \leq 36 \}$.

- If a max or min occurs at an interior point (x, z) of D, then (x, z) must be a critical point of h and hence must obey

$$\begin{aligned} h_x &= 6 + z - 2x = 0 \\ h_z &= x - 2z = 0 \end{aligned}$$

Substituting $x = 2z$ into the first equation gives $6 - 3z = 0$ and hence $z = 2$ and $x = 4$.

- If a max or min occurs a point (x, z) on the boundary of D, we have $x^2 + z^2 = 36$ and hence $x = \pm\sqrt{36 - z^2}$ and $h = 6x + zx = \pm(6 + z)\sqrt{36 - z^2}$ with $-6 \leq z \leq 6$. So the max or min can occur either when $z = -6$ or $z = +6$ or at a z obeying

$$0 = \frac{\mathrm{d}}{\mathrm{d}z}\big[(6 + z)\sqrt{36 - z^2}\big] = \sqrt{36 - z^2} - \frac{z(6 + z)}{\sqrt{36 - z^2}}$$

or equivalently

$$\begin{aligned} 36 - z^2 - z(6+z) &= 0 \\ 2z^2 + 6z - 36 &= 0 \\ z^2 + 3z - 18 &= 0 \\ (z+6)(z-3) &= 0 \end{aligned}$$

So the max or min can occur either when $z = \pm 6$ or $z = 3$.

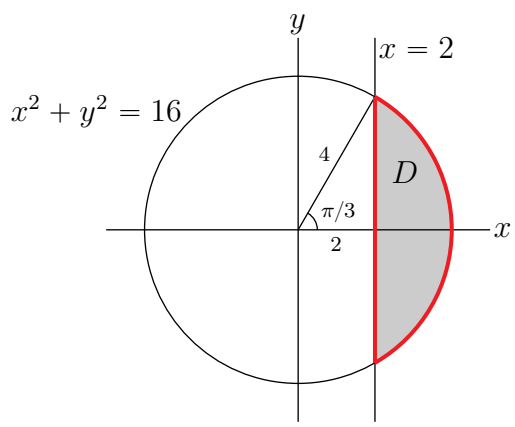
So we have the following candidates for the locations of the min and max

point	$(0, 0, \pm 6)$	$(3\sqrt{3}, 0, 3)$	$(-3\sqrt{3}, 0, 3)$	$(4, 4, 2)$	$(4, -4, 2)$
value of f	0	$27\sqrt{3}$	$-27\sqrt{3}$	48	48
			min	max	max

5. Let D be the region in the xy–plane bounded on the left by the line $x = 2$ and on the right by the circle $x^2 + y^2 = 16$. Evaluate

$$\iint_D \left(x^2 + y^2\right)^{-3/2} \, \mathrm{d}A$$

Solution. Here is a sketch of D.



We'll use polar coordinates. In polar coordinates the circle $x^2 + y^2 = 16$ is $r = 4$ and the line $x = 2$ is $r\cos\theta = 2$. So

$$D = \left\{ (r\cos\theta\,,\, r\sin\theta) \;\middle|\; -\frac{\pi}{3} \le \theta \le \frac{\pi}{3},\ \frac{2}{\cos\theta} \le r \le 4 \right\}$$

and, as $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$, the specified integral is

$$\begin{aligned}
\iint_D \left(x^2+y^2\right)^{-3/2}\,\mathrm{d}A &= \int_{-\pi/3}^{\pi/3} \mathrm{d}\theta \int_{2/\cos\theta}^{4} \mathrm{d}r\; r\frac{1}{r^3} \\
&= \int_{-\pi/3}^{\pi/3} \mathrm{d}\theta\; \left[-\frac{1}{r}\right]_{2/\cos\theta}^{4} \\
&= \int_{-\pi/3}^{\pi/3} \mathrm{d}\theta\; \left[\frac{\cos\theta}{2} - \frac{1}{4}\right] \\
&= \left[\frac{\sin\theta}{2} - \frac{\theta}{4}\right]_{-\pi/3}^{\pi/3} \\
&= \frac{\sqrt{3}}{2} - \frac{\pi}{6}
\end{aligned}$$

6. (a) Let

$$I = \int_0^2 \int_0^x f(x,y)\;\mathrm{d}y\,\mathrm{d}x + \int_2^6 \int_0^{\sqrt{6-x}} f(x,y)\;\mathrm{d}y\,\mathrm{d}x$$

Express I as an integral where we integrate first with respect to x.

(b) Let

$$J = \int_0^1 \int_0^x \int_0^y f(x,y,z)\;\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$$

Express J as an integral where the integrations are to be performed in the order x first, then y, then z.

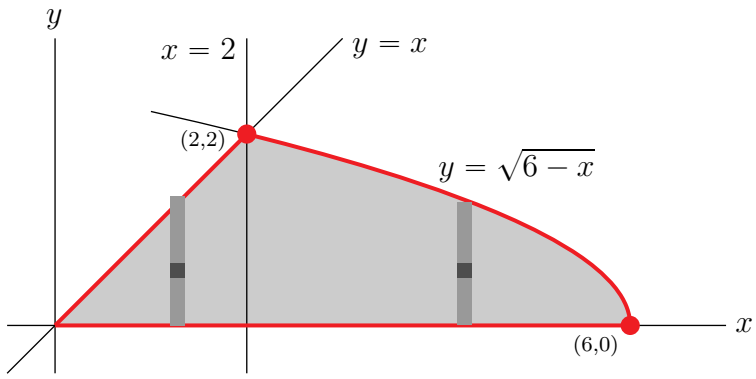
Solution. (a) We first have to get a picture of the domain of integration. The first integral has domain of integration

$$\left\{\ (x,y)\ \middle|\ 0 \le x \le 2,\ 0 \le y \le x\ \right\}$$

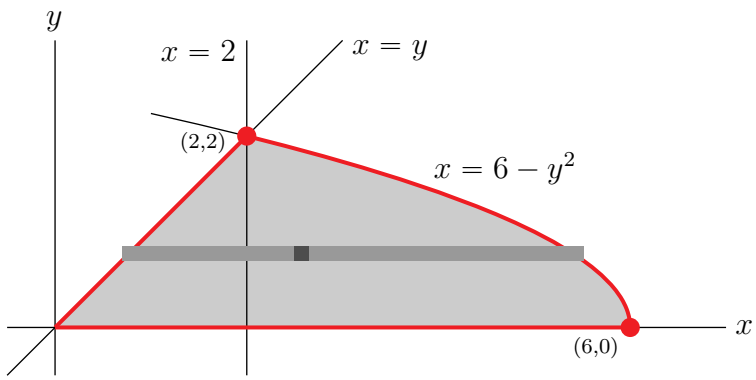
and the second integral has domain of integration

$$\left\{\ (x,y)\ \middle|\ 2 \le x \le 6,\ 0 \le y \le \sqrt{6-x}\ \right\}$$

Here is a sketch. The domain of integration for the first integral is the shaded triangular region to the left of $x = 2$ and the domain of integration for the second integral is the shaded region to the right of $x = 2$.



To exchange the order of integration, we use horizontal slices as in the figure below.



The bottom slice has $y = 0$ and the top slice has $y = 2$. On the slice at height y, x runs from y to $6 - y^2$. So

$$I = \int_0^2 \int_y^{6-y^2} f(x, y)\, \mathrm{d}x\, \mathrm{d}y$$

(b) In the integral J,

- x runs from 0 to 1. In inequalities, $0 \le x \le 1$.
- Then, for each fixed x in that range, y runs from 0 to x. In inequalities, $0 \le y \le x$.
- Then, for each fixed x and y in those ranges, z runs from 0 to y. In inequalities, $0 \le z \le y$.

These inequalties can be combined into

$$0 \le z \le y \le x \le 1 \tag{$*$}$$

We wish to reverse the order of integration so that the z–integral is on the outside, the y–integral is in the middle and the x–integral is on the inside.

- The smallest z compatible with $(*)$ is $z = 0$ and the largest z compatible with $(*)$ is $z = 1$ (when $x = y = z = 1$). So $0 \le z \le 1$.
- Then, for each fixed z in that range, (x, y) run over $z \le y \le x \le 1$. In particular, the smallest allowed y is $y = z$ and the largest allowed y is $y = 1$ (when $x = y = 1$). So $z \le y \le 1$.
- Then, for each fixed y and z in those ranges, x runs over $y \le x \le 1$.

So

$$J = \int_0^1 \int_z^1 \int_y^1 f(x, y, z)\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

7. Let E be the solid lying above the surface $z = y^2$ and below the surface $z = 4 - x^2$. Evaluate

$$\iiint_E y^2\ \mathrm{d}V$$

Hint: you may need to use the half angle formulas:

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}, \qquad \cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$

Solution. Note that the surfaces meet when $z = y^2 = 4 - x^2$ and then (x, y) runs over the circle $x^2 + y^2 = 4$. So the domain of integration is

$$E = \left\{\ (x, y, z)\ \middle|\ x^2 + y^2 \le 4,\ y^2 \le z \le 4 - x^2\ \right\}$$

Let's switch to cylindrical coordinates. Then

$$E = \left\{\ (r\cos\theta, r\sin\theta, z)\ \middle|\ 0 \le r \le 2,\ 0 \le \theta \le 2\pi,\ r^2\sin^2\theta \le z \le 4 - r^2\cos^2\theta\ \right\}$$

and, since $\mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$,

$$\begin{aligned}
\iiint_E y^2\ \mathrm{d}V &= \int_0^2 \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \int_{r^2\sin^2\theta}^{4 - r^2\cos^2\theta} \mathrm{d}z\ r\ \overbrace{r^2\sin^2\theta}^{y^2} \\
&= \int_0^2 \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta\ r^3\sin^2\theta\big[4 - r^2\cos^2\theta - r^2\sin^2\theta\big] \\
&= \int_0^2 \mathrm{d}r\ \big[4r^3 - r^5\big] \int_0^{2\pi} \mathrm{d}\theta\ \frac{1 - \cos(2\theta)}{2} \\
&= \frac{1}{2}\int_0^2 \mathrm{d}r\ \big[4r^3 - r^5\big]\ \left[\theta - \frac{\sin(2\theta)}{2}\right]_0^{2\pi} \\
&= \pi\left[r^4 - \frac{r^6}{6}\right]_0^2 \\
&= \frac{16\pi}{3}
\end{aligned}$$

For an efficient, sneaky, way to evaluate $\int_0^{2\pi} \sin^2\theta \, d\theta$, see Remark 3.3.5 in the CLP–III text.

8. Let E be the solid
$$0 \le z \le \sqrt{x^2+y^2}, \qquad x^2+y^2 \le 1,$$
and consider the integral
$$I = \iiint_E z\sqrt{x^2+y^2+z^2}\, dV.$$

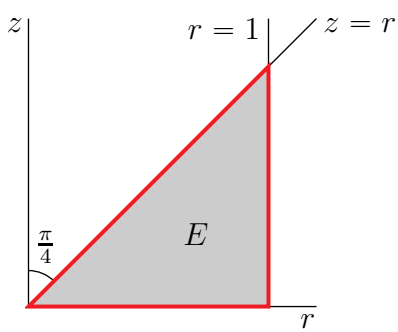
(a) Write the integral I in cylindrical coordinates.

(b) Write the integral I in spherical coordinates.

(c) Evaluate the integral I using either form.

Solution. (a) In cylindrical coordinates $0 \le z \le \sqrt{x^2+y^2}$ becomes $0 \le z \le r$, and $x^2+y^2 \le 1$ becomes $0 \le r \le 1$. So
$$E = \left\{ (r\cos\theta\,,\, r\sin\theta\,,\, z) \;\middle|\; 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le z \le r \right\}$$
and, since $dV = r\,dr\,d\theta\,dz$,
$$I = \iiint_E z\sqrt{x^2+y^2+z^2}\, dV = \int_0^1 dr \int_0^{2\pi} d\theta \int_0^r dz\; r\; z\; \sqrt{\underbrace{r^2+z^2}_{x^2+y^2+z^2}}$$

(b) Here is a sketch of a constant θ section of E.



Recall that the spherical coordinate φ is the angle between the z–axis and the radius vector. So, in spherical coordinates $z = r$ (which makes an angle $\frac{\pi}{4}$ with the z axis) becomes $\varphi = \frac{\pi}{4}$, and the plane $z = 0$, i.e. the xy–plane, becomes $\varphi = \frac{\pi}{2}$, and $r = 1$ becomes $\rho\sin\varphi = 1$. So
$$E = \left\{ (\rho\sin\varphi\cos\theta\,,\, \rho\sin\varphi\sin\theta\,,\, \rho\cos\varphi) \;\middle|\; \frac{\pi}{4} \le \varphi \le \frac{\pi}{2},\ 0 \le \theta \le 2\pi,\ 0 \le \rho \le \frac{1}{\sin\varphi} \right\}$$

and, since $\mathrm{d}V = \rho^2 \sin\varphi \,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi$,

$$\begin{aligned} I &= \iiint_E z\sqrt{x^2+y^2+z^2}\;\mathrm{d}V = \int_{\pi/4}^{\pi/2} \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\theta \int_0^{1/\sin\varphi} \mathrm{d}\rho\;\rho^2 \sin\varphi\;\overbrace{\rho\cos\varphi}^{z}\;\rho \\ &= \int_{\pi/4}^{\pi/2} \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\theta \int_0^{1/\sin\varphi} \mathrm{d}\rho\;\rho^4 \sin\varphi\;\cos\varphi \end{aligned}$$

(c) We'll integrate using the spherical coordinate version.

$$\begin{aligned} I &= \int_{\pi/4}^{\pi/2} \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\theta \int_0^{1/\sin\varphi} \mathrm{d}\rho\;\rho^4 \sin\varphi\;\cos\varphi \\ &= \int_{\pi/4}^{\pi/2} \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\theta\;\frac{1}{5\sin^5\varphi}\sin\varphi\;\cos\varphi \\ &= \frac{2\pi}{5}\int_{\pi/4}^{\pi/2} \mathrm{d}\varphi\;\frac{\cos\varphi}{\sin^4\varphi} \\ &= \frac{2\pi}{5}\int_{1/\sqrt{2}}^{1} \frac{\mathrm{d}u}{u^4} \qquad \text{with } u = \sin\varphi,\ \mathrm{d}u = \cos\varphi\,\mathrm{d}\varphi \\ &= \frac{2\pi}{5}\left[\frac{u^{-3}}{-3}\right]_{1/\sqrt{2}}^{1} \\ &= \frac{2(2\sqrt{2}-1)\pi}{15} \end{aligned}$$