MATHEMATICS 200 April 2011 Final Exam Solutions

1. Let $A = (2, 3, 4)$ and let L be the line given by the equations $x+y = 1$ and $x+2y+z = 3$.

   (a) Write a vector equation for L.

   (b) Write an equation for the plane containing A and perpendicular to L.

   (c) Write an equation for the plane containing A and L.

   Solution. (a) Let's parametrize L using y, renamed to t, as the parameter. Then $y = t$, so that

$$x + y = 1 \implies x + t = 1 \implies x = 1 - t$$

   and

$$x + 2y + z = 3 \implies 1 - t + 2t + z = 3 \implies z = 2 - t$$

   and

$$\langle x, y, z\rangle = \langle 1, 0, 2\rangle + t\,\langle -1, 1, -1\rangle$$

   is a vector parametric equation for L.

   (b) The plane is to contain the point $(2, 3, 4)$ and is to have $\langle -1, 1, -1\rangle$ as a normal vector. So

$$\langle -1, 1, -1\rangle \cdot \langle x - 2, y - 3, z - 4\rangle = 0 \qquad \text{or} \qquad x - y + z = 3$$

   does the job.

   (c) The plane is to contain the points $A = (2, 3, 4)$ and $(1, 0, 2)$ (which is on L) so that the vector $\langle 2 - 1, 3 - 0, 4 - 2\rangle = \langle 1, 3, 2\rangle$ is to be parallel to the plane. The direction vector of L, namely $\langle -1, 1, -1\rangle$, is also to be parallel to the plane. So the vector

$$\langle 1, 3, 2\rangle \times \langle -1, 1, -1\rangle = \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 1 & 3 & 2 \\ -1 & 1 & -1 \end{bmatrix} = \langle -5\,,\ -1\,,\ 4\rangle$$

   is to be normal to the plane. So

$$\langle -5, -1, 4\rangle \cdot \langle x - 2, y - 3, z - 4\rangle = 0 \qquad \text{or} \qquad 5x + y - 4z = -3$$

   does the job.

2. According to van der Waal's equation, a gas satisfies the equation

$$(pV^2 + 16)(V - 1) = TV^2,$$

   where p, V and T denote pressure, volume and temperature respectively. Suppose the gas is now at pressure 1, volume 2 and temperature 5. Find the approximate change in its volume if p is increased by 0.2 and T is increased by 0.3.

Solution. Think of the volume as being the function $V(p,T)$ of pressure and temperature that is determined implicitly (at least for $p \approx 1$, $T \approx 5$ and $V \approx 2$) by the equation

$$\big(pV(p,T)^2 + 16\big)\big(V(p,T) - 1\big) = TV(p,T)^2 \qquad (*)$$

To determine the approximate change in V, we will use the linear approximation to $V(p,T)$ at $p = 1$, $T = 5$. So we will need the partial derivatives $V_p(1,5)$ and $V_T(1,5)$. As the equation $(*)$ is valid for all p near 1 and T near 5, we may differentiate $(*)$ with respect to p, giving

$$\big(V^2 + 2pVV_p\big)\big(V - 1\big) + \big(pV^2 + 16\big)V_p = 2TVV_p$$

and we may also differentiate $(*)$ with respect to T, giving

$$\big(2pVV_T\big)\big(V - 1\big) + \big(pV^2 + 16\big)V_T = V^2 + 2TVV_T$$

In particular, when $p = 1$, $V = 2$, $T = 5$,

$$\begin{aligned} \big(4 + 4V_p(1,5)\big)\big(2-1\big) + \big(4+16\big)V_p(1,5) &= 20V_p(1,5) &&\implies V_p(1,5) = -1 \\ 4V_T(1,5)\big(2-1\big) + \big(4+16\big)V_T(1,5) &= 4 + 20V_T(1,5) &&\implies V_T(1,5) = 1 \end{aligned}$$

so that the change in V is

$$V(1.2\,,\,5.3) - V(1,5) \approx V_p(1,5)\,(0.2) + V_T(1,5)\,(0.3) = -0.2 + 0.3 = 0.1$$

3. Suppose that $w = f(xz, yz)$, where f is a differentiable function. Show that

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = z\frac{\partial w}{\partial z}$$

Solution. We'll first compute the first order partial derivatives of $w(x,y,z)$. By the chain rule,

$$\begin{aligned} \frac{\partial w}{\partial x}(x,y,z) &= \frac{\partial}{\partial x}\big[f(xz,yz)\big] = z\frac{\partial f}{\partial x}(xz,yz) \\ \frac{\partial w}{\partial y}(x,y,z) &= \frac{\partial}{\partial y}\big[f(xz,yz)\big] = z\frac{\partial f}{\partial y}(xz,yz) \\ \frac{\partial w}{\partial z}(x,y,z) &= \frac{\partial}{\partial z}\big[f(xz,yz)\big] = x\frac{\partial f}{\partial x}(xz,yz) + y\frac{\partial f}{\partial y}(xz,yz) \end{aligned}$$

So

$$x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = xz\frac{\partial f}{\partial x}(xz,yz) + yz\frac{\partial f}{\partial y}(xz,yz) = z\left[x\frac{\partial f}{\partial x}(xz,yz) + y\frac{\partial f}{\partial y}(xz,yz)\right] = z\frac{\partial w}{\partial z}$$

as desired.

4. Let

$$f(x,y,z) = (2x+y)e^{-(x^2+y^2+z^2)}$$
$$g(x,y,z) = xz + y^2 + yz + z^2$$

(a) Find the gradients of f and g at $(0,1,-1)$.

(b) A bird at $(0,1,-1)$ flies at speed 6 in the direction in which $f(x,y,z)$ increases most rapidly. As it passes through $(0,1,-1)$, how quickly does $g(x,y,z)$ appear (to the bird) to be changing?

(c) A bat at $(0,1,-1)$ flies in the direction in which $f(x,y,z)$ and $g(x,y,z)$ do not change, but z increases. Find a vector in this direction.

Solution. (a) The first order partial derivatives of f and g are

$$\begin{aligned}
\frac{\partial f}{\partial x}(x,y,z) &= 2e^{-(x^2+y^2+z^2)} - 2x(2x+y)e^{-(x^2+y^2+z^2)} &&\Longrightarrow \frac{\partial f}{\partial x}(0,1,-1) = 2e^{-2}\\
\frac{\partial f}{\partial y}(x,y,z) &= e^{-(x^2+y^2+z^2)} - 2y(2x+y)e^{-(x^2+y^2+z^2)} &&\Longrightarrow \frac{\partial f}{\partial y}(0,1,-1) = -e^{-2}\\
\frac{\partial f}{\partial z}(x,y,z) &= -2z(2x+y)e^{-(x^2+y^2+z^2)} &&\Longrightarrow \frac{\partial f}{\partial z}(0,1,-1) = 2e^{-2}\\
\frac{\partial g}{\partial x}(x,y,z) &= z &&\Longrightarrow \frac{\partial g}{\partial x}(0,1,-1) = -1\\
\frac{\partial g}{\partial y}(x,y,z) &= 2y+z &&\Longrightarrow \frac{\partial g}{\partial y}(0,1,-1) = 1\\
\frac{\partial g}{\partial z}(x,y,z) &= x+y+2z &&\Longrightarrow \frac{\partial g}{\partial z}(0,1,-1) = -1
\end{aligned}$$

so that gradients are

$$\boldsymbol{\nabla} f(0,1,-1) = e^{-2}\langle 2,-1,2\rangle \qquad \boldsymbol{\nabla} g(0,1,-1) = \langle -1,1,-1\rangle$$

(b) The bird's velocity is the vector of length 6 in the direction of $\boldsymbol{\nabla} f(0,1,-1)$, which is

$$\mathbf{v} = 6\frac{\langle 2,-1,2\rangle}{|\langle 2,-1,2\rangle|} = \langle 4,-2,4\rangle$$

The rate of change of g (per unit time) seen by the bird is

$$\boldsymbol{\nabla} g(0,1,-1)\cdot\mathbf{v} = \langle -1,1,-1\rangle\cdot\langle 4,-2,4\rangle = -10$$

(c) The direction of flight for the bat has to be perpendicular to both $\boldsymbol{\nabla} f(0,1,-1) = e^{-2}\langle 2,-1,2\rangle$ and $\boldsymbol{\nabla} g(0,1,-1) = \langle -1,1,-1\rangle$. Any vector which is a non zero constant times

$$\langle 2,-1,2\rangle \times \langle -1,1,-1\rangle = \det\begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ 2 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix} = \langle -1\,,\,0\,,\,1\rangle$$

is perpendicular to both $\boldsymbol{\nabla} f(0,1,-1)$ and $\boldsymbol{\nabla} g(0,1,-1)$. In addition, the direction of flight for the bat must have a positive z–component. So any vector which is a (strictly) positive constant times $\langle -1\,,\,0\,,\,1\rangle$ is fine.

5. Let $h(x,y) = y(4 - x^2 - y^2)$.

   (a) Find and classify the critical points of $h(x,y)$ as local maxima, local minima or saddle points.

   (b) Find the maximum and minimum values of $h(x,y)$ on the disk $x^2 + y^2 \le 1$.

   (c) Find the maximum value of $f(x,y,z) = xyz$ on the ellipsoid

$$g(x,y,z) = x^2 + xy + y^2 + 3z^2 = 9$$

   Specify all points at which this maximum value occurs.

Solution. (a) To find the critical points we will need the gradient of h and to apply the second derivative test of Theorem 2.9.16 in the CLP–III text we will need all second order partial derivatives. So we need all partial derivatives of f up to order two. Here they are.

$$\begin{aligned} h &= y(4 - x^2 - y^2) \\ h_x &= -2xy & h_{xx} &= -2y & h_{xy} &= -2x \\ h_y &= 4 - x^2 - 3y^2 & h_{yy} &= -6y & h_{yx} &= -2x \end{aligned}$$

(Of course, h_{xy} and h_{yx} have to be the same. It is still useful to compute both, as a way to catch some mechanical errors.)

The critical points are the solutions of

$$h_x = -2xy = 0 \qquad h_y = 4 - x^2 - 3y^2 = 0$$

The first equation is satisfied if at least one of $x = 0$, $y = 0$ are satisfied.

- If $x = 0$, the second equation reduces to $4 - 3y^2 = 0$, which is satisfied if $y = \pm\frac{2}{\sqrt{3}}$.
- If $y = 0$, the second equation reduces to $4 - x^2 = 0$ which is satisfied if $x = \pm 2$.

So there are four critical points: $\left(0, \frac{2}{\sqrt{3}}\right)$, $\left(0, -\frac{2}{\sqrt{3}}\right)$, $(2,0)$, $(-2,0)$.

The classification is

critical point	$h_{xx}h_{yy} - h_{xy}^2$	h_{xx}	type
$\left(0, \frac{2}{\sqrt{3}}\right)$	$\left(\frac{-4}{\sqrt{3}}\right) \times \left(-\frac{12}{\sqrt{3}}\right) - (0)^2 > 0$	$\frac{-4}{\sqrt{3}}$	local max
$\left(0, -\frac{2}{\sqrt{3}}\right)$	$\left(\frac{4}{\sqrt{3}}\right) \times \left(\frac{12}{\sqrt{3}}\right) - (0)^2 > 0$	$\frac{4}{\sqrt{3}}$	local min
$(2,0)$	$0 \times 0 - (-4)^2 < 0$		saddle point
$(-2,0)$	$0 \times 0 - (4)^2 < 0$		saddle point

(b) The absolute max and min can occur either in the interior of the disk or on the boundary of the disk. The boundary of the disk is the circle $x^2 + y^2 = 1$.

- Any absolute max or min in the interior of the disk must also be a local max or min and, since there are no singular points, must also be a critical point of h. We found all of the critical points of h in part (a). Since $2 > 1$ and $\frac{2}{\sqrt{3}} > 1$ none of the critical points are in the disk.
- At each point of $x^2 + y^2 = 1$ we have $h(x, y) = 3y$ with $-1 \le y \le 1$. Clearly the maximum value is 3 (at $(0, 1)$) and the minimum value is -3 (at $(0, -1)$).

So all together, the maximum and minimum values of $h(x, y)$ in $x^2 + y^2 \le 1$ are 3 (at $(0, 1)$) and -3 (at $(0, -1)$), respectively.

(c) This is a constrained optimization problem with the objective function being

$$f(x, y, z) = xyz$$

and the constraint function being

$$G(x, y, z) = x^2 + xy + y^2 + 3z^2 - 9$$

By Theorem 2.10.2 in the CLP–III text, any local minimum or maximum (x, y, z) must obey the Lagrange multiplier equations

$$f_x = yz = \lambda(2x + y) = \lambda G_x \tag{E1}$$
$$f_y = xz = \lambda(2y + x) = \lambda G_y \tag{E2}$$
$$f_z = xy = 6\lambda z = \lambda G_z \tag{E3}$$
$$x^2 + xy + y^2 + 3z^2 = 9 \tag{E4}$$

for some real number λ.

- If $\lambda = 0$, then, by (E1), $yz = 0$ so that $f(x, y, z) = xyz = 0$. This cannot possibly be the maximum value of f because there are points (x, y, z) on $g(x, y, z) = 9$ (for example $x = y = 1$, $z = \sqrt{2}$) with $f(x, y, z) > 0$.
- If $\lambda \ne 0$, then multiplying (E1) by x, (E2) by y, and (E3) by z gives

$$\begin{aligned} xyz = \lambda(2x^2 + xy) = \lambda(2y^2 + xy) = 6\lambda z^2 &\implies 2x^2 + xy = 2y^2 + xy = 6z^2 \\ &\implies x = \pm y,\ z^2 = \frac{1}{6}(2x^2 + xy) \end{aligned}$$

  - If $x = y$, then $z^2 = \frac{x^2}{2}$ and, by (E4)

$$x^2 + x^2 + x^2 + \frac{3}{2}x^2 = 9 \implies x^2 = 2 \implies x = y = \pm\sqrt{2},\ z = \pm 1$$

    For these points

$$f(x, y, z) = 2z = \begin{cases} 2 & \text{if } z = 1 \\ -2 & \text{if } z = -1 \end{cases}$$

– If $x = -y$, then $z^2 = \frac{x^2}{6}$ and, by (E4)

$$x^2 - x^2 + x^2 + \frac{x^2}{2} = 9 \implies x^2 = 6 \implies x = -y = \pm\sqrt{6},\ z = \pm 1$$

For these points

$$f(x, y, z) = -6z = \begin{cases} -6 & \text{if } z = 1 \\ 6 & \text{if } z = -1 \end{cases}$$

So the maximum is 6 and is achieved at $\left(\sqrt{6}, -\sqrt{6}, -1\right)$ and $\left(-\sqrt{6}, \sqrt{6}, -1\right)$.

6. Consider

$$J = \int_0^{\sqrt{2}} \int_y^{\sqrt{4-y^2}} \frac{y}{x} e^{x^2+y^2}\, \mathrm{d}x\, \mathrm{d}y$$

(a) Sketch the region of integration.

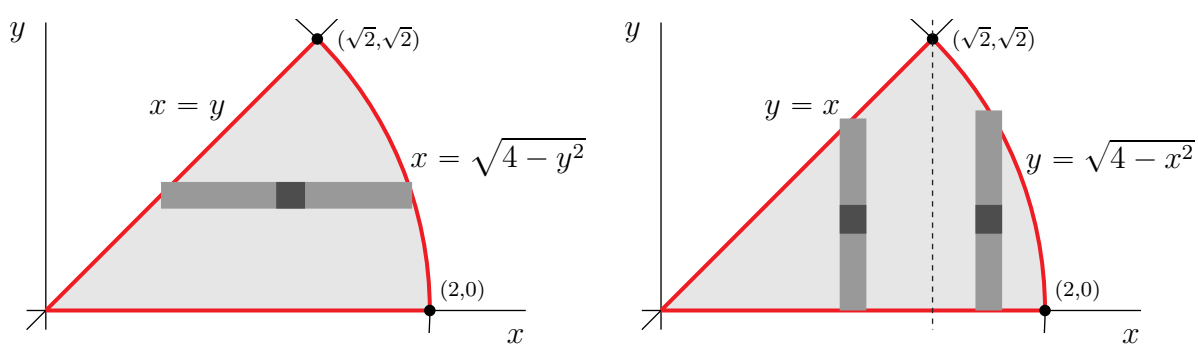
(b) Reverse the order of integration.

(c) Evaluate J by using polar coordinates.

Solution. (a) On the domain of integration

- y runs from 0 to $\sqrt{2}$ and
- for each y in that range, x runs from y to $\sqrt{4-y^2}$. We can rewrite $x = \sqrt{4-y^2}$ in the more familiar form $x^2 + y^2 = 4$, $x \geq 0$.

The figure on the left below provides a sketch of the domain of integration. It also shows the generic horizontal slice that was used to set up the given iterated integral.



(b) To reverse the order of integration observe, we use vertical, rather than horizontal slices. From the figure on the right above that, on the domain of integration,

- x runs from 0 to 2 and
- for each x in the range $0 \leq x \leq \sqrt{2}$, y runs from 0 to x.

- for each x in the range $\sqrt{2} \le x \le 2$, y runs from 0 to $\sqrt{4-x^2}$.

So the integral

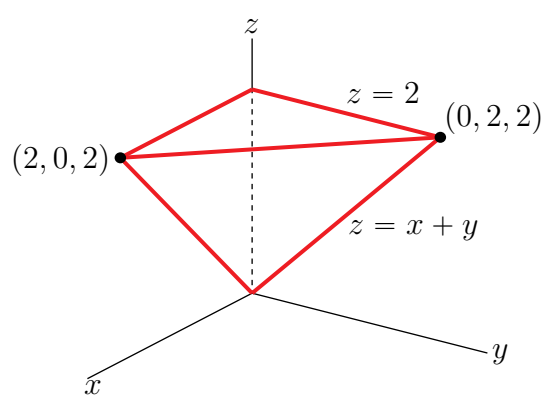
$$J = \int_0^{\sqrt{2}} \int_0^x \frac{y}{x} e^{x^2+y^2} \, \mathrm{d}y \, \mathrm{d}x + \int_{\sqrt{2}}^2 \int_0^{\sqrt{4-x^2}} \frac{y}{x} e^{x^2+y^2} \, \mathrm{d}y \, \mathrm{d}x$$

(c) In polar coordinates, the line $y = x$ is $\theta = \frac{\pi}{4}$, the circle $x^2 + y^2 = 4$ is $r = 2$, and $\mathrm{d}x \, \mathrm{d}y = r \, \mathrm{d}r \, \mathrm{d}\theta$. So

$$\begin{aligned}
J &= \int_0^{\pi/4} \mathrm{d}\theta \int_0^2 \mathrm{d}r \; r \overbrace{\frac{r \sin\theta}{r \cos\theta}}^{\frac{y}{x}} e^{r^2} \\
&= \int_0^{\pi/4} \mathrm{d}\theta \; \frac{\sin\theta}{\cos\theta} \left[\frac{1}{2} e^{r^2} \right]_0^2 \\
&= -\frac{1}{2} \left[e^4 - 1 \right] \int_1^{1/\sqrt{2}} \mathrm{d}u \; \frac{1}{u} \qquad \text{with } u = \cos\theta, \; \mathrm{d}u = -\sin\theta \, \mathrm{d}\theta \\
&= -\frac{1}{2} \left[e^4 - 1 \right] \; \Big[\ln|u| \Big]_1^{1/\sqrt{2}} \\
&= \frac{1}{4} \left[e^4 - 1 \right] \ln 2
\end{aligned}$$

7. Let E be the portion of the first octant which is above the plane $z = x + y$ and below the plane $z = 2$. The density in E is $\rho(x, y, z) = z$. Find the mass of E.

Solution. Note that the planes $z = x + y$ and $z = 2$ intersect along the line $x + y = 2$, $z = 2$.



So

$$\begin{aligned}
E &= \left\{ \; (x, y, z) \;\middle|\; x \ge 0, \; y \ge 0, \; x + y \le 2, \; x + y \le z \le 2 \; \right\} \\
&= \left\{ \; (x, y, z) \;\middle|\; 0 \le x \le 2, \; 0 \le y \le 2 - x, \; x + y \le z \le 2 \; \right\}
\end{aligned}$$

and the mass of E is

$$\begin{aligned}
\iiint_E \rho(x,y,z)\,\mathrm{d}V &= \int_0^2 \mathrm{d}x \int_0^{2-x} \mathrm{d}y \int_{x+y}^2 \mathrm{d}z\ z \\
&= \frac{1}{2}\int_0^2 \mathrm{d}x \int_0^{2-x} \mathrm{d}y\ \left[4-(x+y)^2\right] \\
&= \frac{1}{2}\int_0^2 \mathrm{d}x\ \left[4(2-x) - \frac{\big(x+(2-x)\big)^3 - x^3}{3}\right] \\
&= \frac{1}{2}\left[4(2)(2) - 2(2)^2 - \frac{8}{3}(2) + \frac{2^4}{12}\right] = \frac{1}{2}\left[8 - \frac{16}{3} + \frac{4}{3}\right] \\
&= 2
\end{aligned}$$

8. Let E be the "ice cream cone" $x^2+y^2+z^2 \le 1$, $x^2+y^2 \le z^2$, $z \ge 0$. Consider

$$J = \iiint_E \sqrt{x^2+y^2+z^2}\ \mathrm{d}V$$

(a) Write J as an iterated integral, with limits, in cylindrical coordinates.

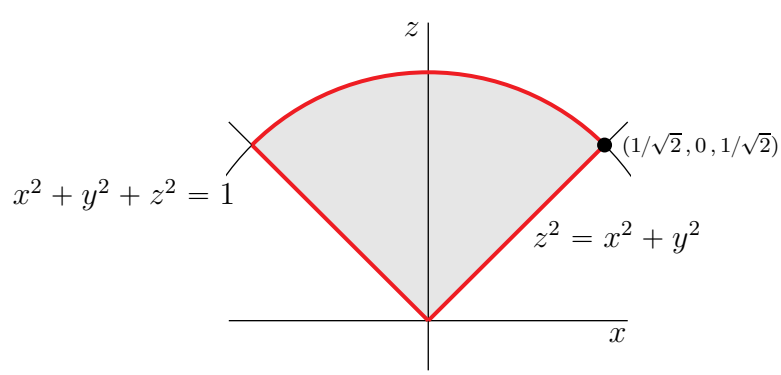
(b) Write J as an iterated integral, with limits, in spherical coordinates.

(c) Evaluate J.

Solution. (a) In cylindrical coordinates

- $x^2+y^2+z^2 \le 1$ is $r^2+z^2 \le 1$ and
- $x^2+y^2 \le z^2$ is $r^2 \le z^2$ and
- $\mathrm{d}V$ is $r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$

Observe that $r^2+z^2=1$ and $r^2=z^2$ intersect when $r^2=z^2=\frac{1}{2}$. Here is a sketch of the $y=0$ cross–section of E.



So

$$E = \left\{ \ (r\cos\theta\,,\ r\sin\theta\,,\ z) \ \middle|\ 0 \le r \le {}^1\!/\!\sqrt{2},\ 0 \le \theta \le 2\pi,\ r \le z \le \sqrt{1-r^2} \ \right\}$$

and

$$J = \iiint_E \sqrt{x^2+y^2+z^2}\ \mathrm{d}V = \int_0^{1/\sqrt{2}} \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \int_r^{\sqrt{1-r^2}} \mathrm{d}z\ r\sqrt{r^2+z^2}$$

(b) In spherical coordinates

- $x^2+y^2+z^2 \le 1$ is $\rho \le 1$ and
- $x^2+y^2 \le z^2$ is $\rho^2\sin^2\varphi \le \rho^2\cos^2\varphi$, or $\tan\varphi \le 1$ or $\varphi \le \frac{\pi}{4}$, and
- $\mathrm{d}V$ is $\rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi$

So

$$E = \left\{ \ (\rho\sin\varphi\cos\theta\,,\ \rho\sin\varphi\sin\theta\,,\ \rho\cos\varphi) \ \middle|\ 0 \le \rho \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le {}^\pi\!/_4 \ \right\}$$

and, since the integrand $\sqrt{x^2+y^2+z^2} = \rho$,

$$\begin{aligned} J &= \iiint_E \sqrt{x^2+y^2+z^2}\ \mathrm{d}V = \int_0^1 \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi/4} \mathrm{d}\varphi\ \rho^2\sin\varphi\ \rho \\ &= \int_0^1 \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi/4} \mathrm{d}\varphi\ \rho^3\sin\varphi \end{aligned}$$

(c) We'll use the spherical coordinate form to evaluate

$$\begin{aligned} J &= \int_0^1 \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi/4} \mathrm{d}\varphi\ \rho^3\sin\varphi \\ &= 2\pi \int_0^1 \mathrm{d}\rho\ \rho^3 \Big[-\cos\varphi \Big]_0^{\pi/4} = 2\pi\ \frac{1}{4}\ \left[1 - \frac{1}{\sqrt{2}}\right] \\ &= \frac{\pi}{2}\ \left[1 - \frac{1}{\sqrt{2}}\right] \end{aligned}$$