MATHEMATICS 200 April 2010 Final Exam Solutions

1. (a) A surface $z(x,y)$ is defined by $zy - y + x = \ln(xyz)$.
   (i) Compute $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ in terms of x, y, z.
   (ii) Evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(x,y,z) = (-1,-2,1/2)$.

   (b) A surface $z = f(x,y)$ has derivatives $\frac{\partial f}{\partial x} = 3$ and $\frac{\partial f}{\partial y} = -2$ at $(x,y,z) = (1,3,1)$.
   (i) If x increases from 1 to 1.2, and y decreases from 3 to 2.6, find the change in z using a linear approximation.
   (ii) Find the equation of the tangent plane to the surface at the point $(1,3,1)$.

Solution. (a) (i) We are told that $z(x,y)$ obeys

$$z(x,y)\,y - y + x = \ln\big(xy\,z(x,y)\big) \qquad (*)$$

for all (x,y) (near $(-1,-2)$). Differentiating $(*)$ with respect to x gives

$$y\,\frac{\partial z}{\partial x}(x,y) + 1 = \frac{1}{x} + \frac{\frac{\partial z}{\partial x}(x,y)}{z(x,y)} \implies \frac{\partial z}{\partial x}(x,y) = \frac{\frac{1}{x} - 1}{y - \frac{1}{z(x,y)}}$$

or, dropping the arguments (x,y) and multiplying both the numerator and denominator by xz,

$$\frac{\partial z}{\partial x} = \frac{z - xz}{xyz - x} = \frac{z(1-x)}{x(yz-1)}$$

Differentiating $(*)$ with respect to y gives

$$z(x,y) + y\,\frac{\partial z}{\partial y}(x,y) - 1 = \frac{1}{y} + \frac{\frac{\partial z}{\partial y}(x,y)}{z(x,y)} \implies \frac{\partial z}{\partial y}(x,y) = \frac{\frac{1}{y} + 1 - z(x,y)}{y - \frac{1}{z(x,y)}}$$

or, dropping the arguments (x,y) and multiplying both the numerator and denominator by yz,

$$\frac{\partial z}{\partial y} = \frac{z + yz - yz^2}{y^2 z - y} = \frac{z(1 + y - yz)}{y(yz-1)}$$

(a) (ii) When $(x,y,z) = (-1,-2,1/2)$,

$$\frac{\partial z}{\partial x}(-1,-2) = \left.\frac{\frac{1}{x} - 1}{y - \frac{1}{z}}\right|_{(x,y,z)=(-1,-2,1/2)} = \frac{\frac{1}{-1} - 1}{-2-2} = \frac{1}{2}$$

$$\frac{\partial z}{\partial y}(-1,-2) = \left.\frac{\frac{1}{y} + 1 - z}{y - \frac{1}{z}}\right|_{(x,y,z)=(-1,-2,1/2)} = \frac{\frac{1}{-2} + 1 - \frac{1}{2}}{-2-2} = 0$$

(b) (i) The linear approximation to $f(x, y)$ at $(1, 3)$ is

$$f(x, y) \approx f(1, 3) + f_x(1, 3)\,(x - 1) + f_y(1, 3)\,(y - 3) = 1 + 3(x - 1) - 2(y - 3)$$

So the change is z is approximately

$$3(1.2 - 1) - 2(2.6 - 3) = 1.4$$

(ii) The equation of the tangent plane is

$$z = f(1, 3) + f_x(1, 3)\,(x - 1) + f_y(1, 3)\,(y - 3) = 1 + 3(x - 1) - 2(y - 3)$$

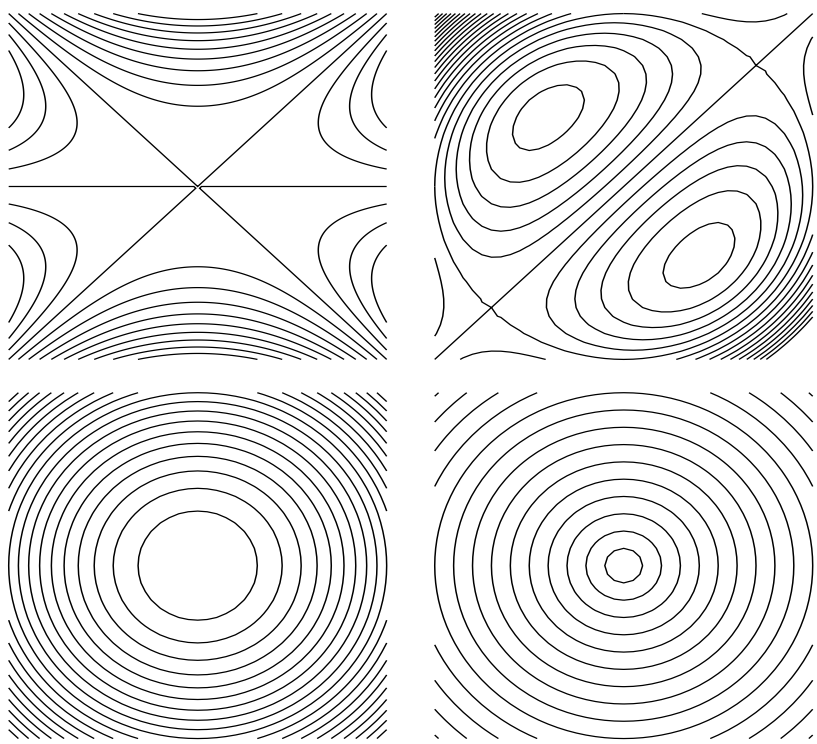
or

$$3x - 2y - z = -4$$

2. (a) For the function $z = f(x, y) = x^3 + 3xy + 3y^2 - 6x - 3y - 6$. Find and classify as [local maxima, local minima, or saddle points] all critical points of $f(x, y)$.

(b) The images below depict level sets $f(x, y) = c$ of the functions in the list at heights $c = 0, 0.1, 0.2, \ldots, 1.9, 2$. Label the pictures with the corresponding function and mark the critical points in each picture. (Note that in some cases, the critical points might not be drawn on the images already. In those cases you should add them to the picture.)

(i) $f(x, y) = (x^2 + y^2 - 1)(x - y) + 1$
(ii) $f(x, y) = \sqrt{x^2 + y^2}$
(iii) $f(x, y) = y(x + y)(x - y) + 1$
(iv) $f(x, y) = x^2 + y^2$



Solution. (a) To find the critical points we will need the gradient of f and to apply the second derivative test of Theorem 2.9.16 in the CLP–III text we will need all second order partial derivatives. So we need all partial derivatives of f up to order two. Here they are.

$$\begin{aligned} f &= x^3 + 3xy + 3y^2 - 6x - 3y - 6 \\ f_x &= 3x^2 + 3y - 6 \qquad f_{xx} = 6x \qquad f_{xy} = 3 \\ f_y &= 3x + 6y - 3 \qquad f_{yy} = 6 \qquad f_{yx} = 3 \end{aligned}$$

The critical points are the solutions of

$$f_x = 3x^2 + 3y - 6 = 0 \qquad f_y = 3x + 6y - 3 = 0$$

Subtracting the second equation from 2 times the first equation gives

$$6x^2 - 3x - 9 = 0 \iff 3(2x-3)(x+1) = 0 \iff x = \frac{3}{2},\ -1$$

Since $y = \frac{1-x}{2}$ (from the second equation), the critical points are $(\frac{3}{2}, -\frac{1}{4})$, $(-1, 1)$ and the classification is

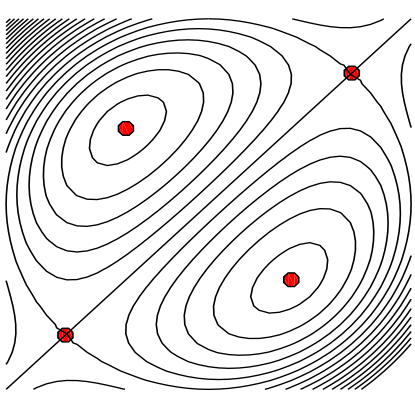
critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(\frac{3}{2}, -\frac{1}{4})$	$(9) \times (6) - (3)^2 > 0$	9	local min
$(-1, 1)$	$(-6) \times (6) - (3)^2 < 0$		saddle point

(b) Both of the functions $f(x,y) = \sqrt{x^2+y^2}$ (i.e. (ii)) and $f(x,y) = x^2 + y^2$ (i.e. (iv)) are invariant under rotations around the $(0,0)$. So their level curves are circles centred on the origin. In polar coordinates $\sqrt{x^2+y^2}$ is r. So the sketched level curves of the function in (ii) are $r = 0, 0.1, 0.2, \ldots, 1.9, 2$. They are equally spaced. So at this point, we know that the third picture goes with (iv) and the fourth picture goes with (ii).

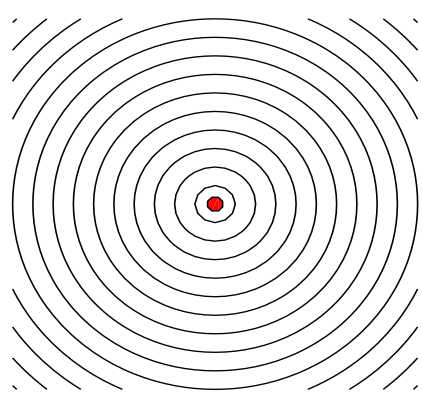
Notice that the lines $x = y$, $x = -y$ and $y = 0$ are all level curves of the function $f(x,y) = y(x+y)(x-y) + 1$ (i.e. of (iii)) with $f = 1$. So the first picture goes with (iii). And the second picture goes with (i).

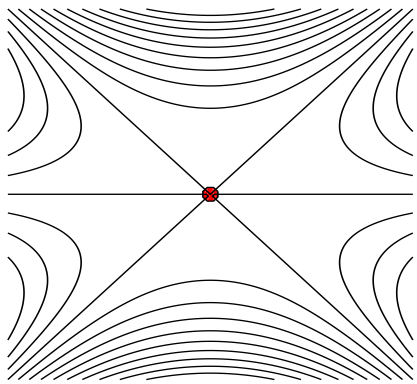
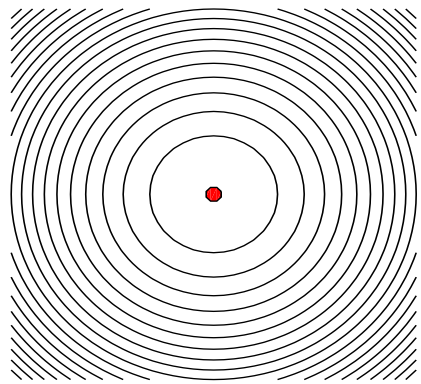
Here are the pictures with critical points marked on them. There are saddle points where level curves cross and there are local max's or min's at "bull's eyes".

(i)



(ii)



(iii)  (iv) 

3. The temperature $T(x, y)$ at a point of the xy–plane is given by

$$T(x, y) = ye^{x^2}$$

A bug travels from left to right along the curve $y = x^2$ at a speed of 0.01m/sec. The bug monitors $T(x, y)$ continuously. What is the rate of change of T as the bug passes through the point $(1, 1)$?

Solution. The slope of $y = x^2$ at $(1, 1)$ is $\frac{\mathrm{d}}{\mathrm{d}x}x^2\Big|_{x=1} = 2$. So a unit vector in the bug's direction of motion is $\frac{\langle 1,2\rangle}{\sqrt{5}}$ and the bug's velocity vector is $\mathbf{v} = 0.01\frac{\langle 1,2\rangle}{\sqrt{5}}$.

The temperature gradient at $(1, 1)$ is

$$\boldsymbol{\nabla} T(1, 1) = \left\langle 2xye^{x^2}\,,\, e^{x^2}\right\rangle\Big|_{(x.y)=(1,1)} = \langle 2e\,,\, e\rangle$$

and the rate of change of T (per unit time) that the bug feels as it passes through the point $(1, 1)$ is

$$\boldsymbol{\nabla} T(1, 1) \cdot \mathbf{v} = \frac{0.01}{\sqrt{5}}\,\langle 2e\,,\, e\rangle \cdot \langle 1, 2\rangle = \frac{0.04e}{\sqrt{5}}$$

4. (a) Does the function $f(x, y) = x^2 + y^2$ have a maximum or a minimum on the curve $xy = 1$? Explain.

   (b) Find all maxima and minima of $f(x, y)$ on the curve $xy = 1$.

Solution. (a) $f(x, y) = x^2 + y^2$ is the square of the distance from the point (x, y) to the origin. There are points on the curve $xy = 1$ that have either x or y arbitrarily large and so whose distance from the origin is arbitrarily large. So f has no maximum on the curve. On the other hand f will have a minimum, achieved at the points of $xy = 1$ that are closest to the origin.

(b) On the curve $xy = 1$ we have $y = \frac{1}{x}$ and hence $f = x^2 + \frac{1}{x^2}$. As

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 + \frac{1}{x^2}\right) = 2x - \frac{2}{x^3} = \frac{2}{x^3}(x^4 - 1)$$

and as no point of the curve has $x = 0$, the minimum is achieved when $x = \pm 1$. So the minima are at $\pm(1, 1)$, where f takes the value 2.

5. Let G be the region in $\mathbb{R}^2$ given by

$$\begin{gathered} x^2 + y^2 \leq 1 \\ 0 \leq x \leq 2y \\ y \leq 2x \end{gathered}$$

(a) Sketch the region G.

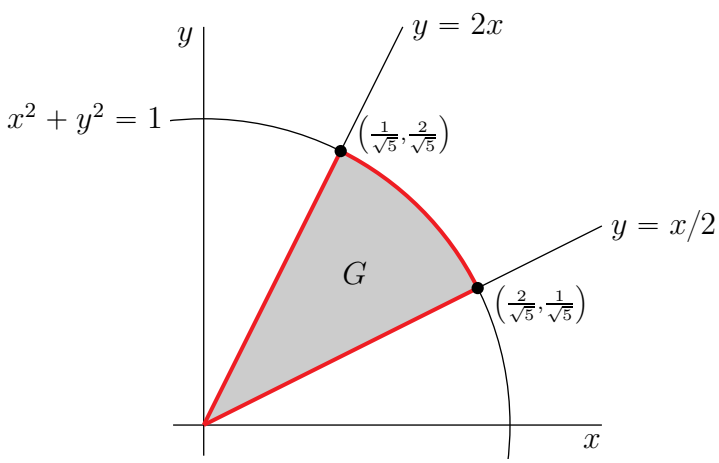
(b) Express the integral $\iint_G f(x, y)\, \mathrm{d}A$ a sum of iterated integrals $\iint f(x, y)\, \mathrm{d}x\mathrm{d}y$.

(c) Express the integral $\iint_G f(x, y)\, \mathrm{d}A$ as an iterated integral in polar coordinates (r, θ) where $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

Solution. (a) Observe that

- the condition $x^2 + y^2 \leq 1$ restricts G to the interior of the circle of radius 1 centred on the origin, and
- the conditions $0 \leq x \leq 2y$ restricts G to $x \geq 0$, $y \geq 0$, i.e. to the first quadrant, and
- the conditions $x \leq 2y$ and $y \leq 2x$ restrict $\frac{x}{2} \leq y \leq 2x$. So G lies below the (steep) line $y = 2x$ and lies above the (not steep) line $y = \frac{x}{2}$.

Here is a sketch of G



(b) Observe that the line $y = 2x$ crosses the circle $x^2 + y^2 = 1$ at a point (x, y) obeying

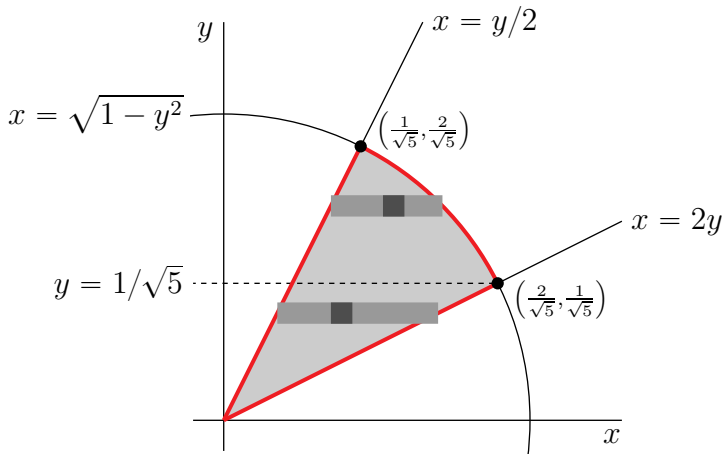
$$x^2 + (2x)^2 = x^2 + y^2 = 1 \implies 5x^2 = 1$$

and that the line $x = 2y$ crosses the circle $x^2 + y^2 = 1$ at a point (x, y) obeying

$$(2y)^2 + y^2 = x^2 + y^2 = 1 \implies 5y^2 = 1$$

So the intersection point of $y = 2x$ and $x^2 + y^2 = 1$ in the first octant is $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ and the intersection point of $x = 2y$ and $x^2 + y^2 = 1$ in the first octant is $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$.

We'll set up the iterated integral using horizontal strips as in the sketch



Looking at that sketch, we see that, on G,

- y runs from 0 to $\frac{2}{\sqrt{5}}$, and
- for each fixed y between 0 and $\frac{1}{\sqrt{5}}$, x runs from $\frac{y}{2}$ to $2y$, and
- for each fixed y between $\frac{1}{\sqrt{5}}$ and $\frac{2}{\sqrt{5}}$ x runs from $\frac{y}{2}$ to $\sqrt{1-y^2}$.

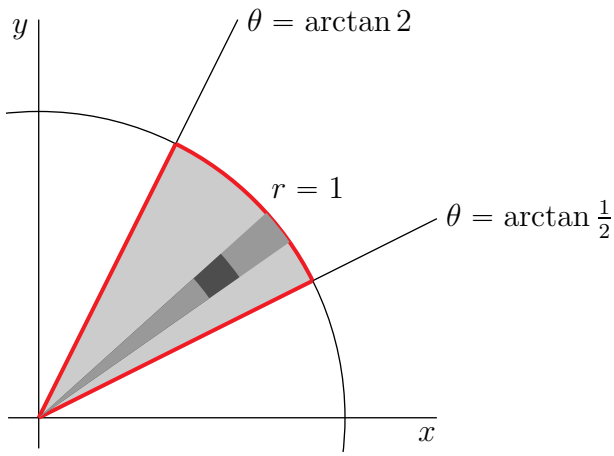
So

$$\iint_G f(x,y)\ \mathrm{d}A = \int_0^{\frac{1}{\sqrt{5}}} \mathrm{d}y \int_{y/2}^{2y} \mathrm{d}x\ f(x,y) + \int_{\frac{1}{\sqrt{5}}}^{\frac{2}{\sqrt{5}}} \mathrm{d}y \int_{y/2}^{\sqrt{1-y^2}} \mathrm{d}x\ f(x,y)$$

(b) In polar coordinates

- the equation $x^2 + y^2 = 1$ becomes $r = 1$, and
- the equation $y = x/2$ becomes $r\sin\theta = \frac{r}{2}\cos\theta$ or $\tan\theta = \frac{1}{2}$, and
- the equation $y = 2x$ becomes $r\sin\theta = 2r\cos\theta$ or $\tan\theta = 2$.

Looking at the sketch



we see that, on G,

- θ runs from $\arctan \frac{1}{2}$ to $\arctan 2$, and
- for each fixed θ in that range, r runs from 0 to 1.

As $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$, and $x = r\cos\theta$, $y = r\sin\theta$,

$$\iint_G f(x,y)\ \mathrm{d}A = \int_{\arctan\frac{1}{2}}^{\arctan 2} \mathrm{d}\theta \int_0^1 \mathrm{d}r\ r\, f(r\cos\theta, r\sin\theta)$$

6. For the integral

$$I = \int_0^1 \int_{\sqrt{x}}^1 \sqrt{1+y^3}\ \mathrm{d}y\,\mathrm{d}x$$

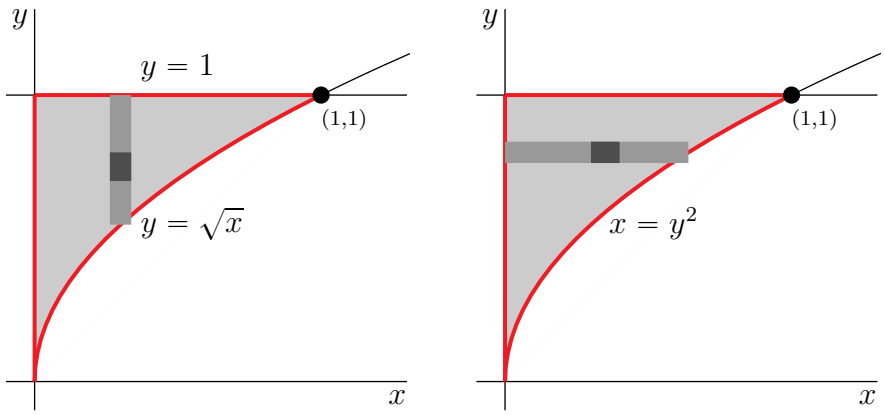
(a) Sketch the region of integration.

(b) Evaluate I.

Solution. (a) On the domain of integration,

- x runs from 0 to 1, and
- for each fixed x in that range, y runs from $\sqrt{x}$ to 1. We may rewrite $y = \sqrt{x}$ as $x = y^2$, which is a rightward opening parabola.

Here are two sketches of the domain of integration, which we call D. The left hand sketch also shows a vertical slice, as was used in setting up the integral.



(b) The inside integral, $\int_{\sqrt{x}}^{1} \sqrt{1+y^3}\ \mathrm{d}y$, of the given integral looks pretty nasty. So let's reverse the order of integration, by using horizontal, rather than vertical, slices. Looking at the figure on the right above, we see that

- y runs from 0 to 1, and
- for each fixed y in that range x runs from 0 to y^2.

So

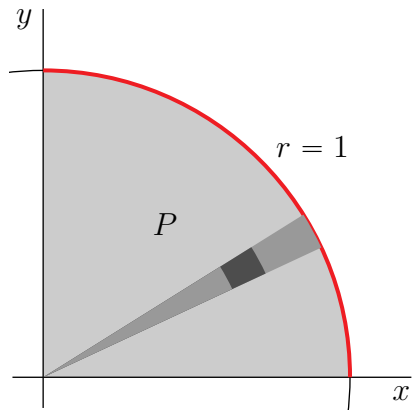
$$\begin{aligned}
I &= \int_0^1 \mathrm{d}y \int_0^{y^2} \mathrm{d}x\ \sqrt{1+y^3} \\
&= \int_0^1 \mathrm{d}y\ y^2 \sqrt{1+y^3} \\
&= \int_1^2 \frac{\mathrm{d}u}{3}\ \sqrt{u} \qquad \text{with } u = 1+y^3,\ \mathrm{d}u = 3y^2\,\mathrm{d}y. \text{ Looks pretty rigged!} \\
&= \frac{1}{3}\left[\frac{u^{3/2}}{3/2}\right]_1^2 \\
&= \frac{2\big(2\sqrt{2}-1\big)}{9}
\end{aligned}$$

7. A thin plate of uniform density k is bounded by the positive x and y axes and the circle $x^2+y^2=1$. Find its centre of mass.

Solution. Call the plate P. By definition, the centre of mass is $(\bar{x},\bar{y})$, with $\bar{x}$ and $\bar{y}$ being the weighted averages of the x and y–coordinates, respectively, over P. That is,

$$\bar{x} = \frac{\iint_P x\ \rho(x,y)\ \mathrm{d}A}{\iint_P \rho(x,y)\ \mathrm{d}A} \qquad \bar{y} = \frac{\iint_P y\ \rho(x,y)\ \mathrm{d}A}{\iint_P \rho(x,y)\ \mathrm{d}A}$$

with $\rho(x,y)=k$. Here is a sketch of P.



By symmetry under reflection in the line $y = x$, we have $\bar{y} = \bar{x}$. So we just have to determine

$$\bar{x} = \frac{\iint_P x \, \mathrm{d}A}{\iint_P \mathrm{d}A}$$

The denominator is just one quarter of the area of circular disk of radius 1. That is, $\iint_P \mathrm{d}A = \frac{\pi}{4}$. We'll evaluate the numerator using polar coordinates as in the figure above. Looking at that figure, we see that

- θ runs from 0 to $\frac{\pi}{2}$, and
- for each fixed θ in that range, r runs from 0 to 1.

As $\mathrm{d}A = r \, \mathrm{d}r \, \mathrm{d}\theta$, and $x = r\cos\theta$, the numerator

$$\begin{aligned}
\iint_P x \, \mathrm{d}A &= \int_0^{\pi/2} \mathrm{d}\theta \int_0^1 \mathrm{d}r \; r \overbrace{r\cos\theta}^{x} = \left[\int_0^{\pi/2} \mathrm{d}\theta \;\; \cos\theta\right] \left[\int_0^1 \mathrm{d}r \; r^2\right] \\
&= \Big[\sin\theta\Big]_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^1 \\
&= \frac{1}{3}
\end{aligned}$$

All together

$$\bar{x} = \bar{y} = \frac{1/3}{\pi/4} = \frac{4}{3\pi}$$

8. Let

$$I = \iiint_T (x^2 + y^2) \, \mathrm{d}V$$

where T is the solid region bounded below by the cone $z = \sqrt{3x^2 + 3y^2}$ and above by the sphere $x^2 + y^2 + z^2 = 9$.

(a) Express I as a triple integral in spherical coordinates.
(b) Express I as a triple integral in cylindrical coordinates.
(c) Evaluate I by any method.

Solution. (a) Recall that in spherical coordinates

$$x = \rho\,\sin\varphi\,\cos\theta \qquad y = \rho\,\sin\varphi\,\sin\theta \qquad z = \rho\,\cos\varphi$$

so that

- $x^2 + y^2 + z^2 \le 9$ is $\rho \le 3$, and
- $\sqrt{3x^2 + 3y^2} \le z$ is $\sqrt{3}\rho\sin\varphi \le \rho\cos\varphi$, or $\tan\varphi \le \frac{1}{\sqrt{3}}$ or $\varphi \le \frac{\pi}{6}$, and
- the integrand $x^2 + y^2 = \rho^2\sin^2\varphi$, and
- $\mathrm{d}V$ is $\rho^2\sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi$

So

$$T = \left\{\ (\rho\sin\varphi\cos\theta\,,\ \rho\sin\varphi\sin\theta\,,\ \rho\cos\varphi)\ \middle|\ 0 \le \rho \le 3,\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi/6\ \right\}$$

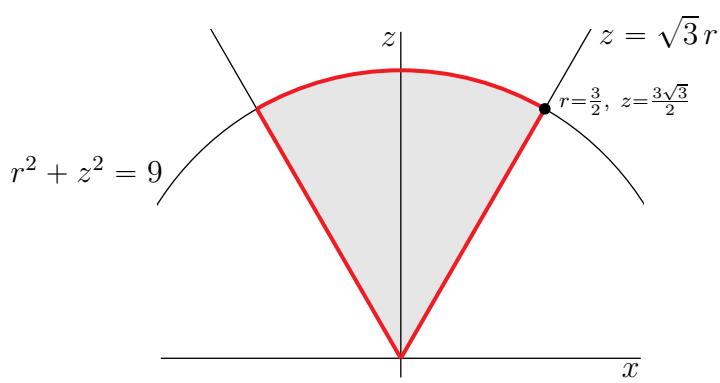
and,

$$\begin{aligned} I &= \iiint_T \left(x^2 + y^2\right)\,\mathrm{d}V = \int_0^3 \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi/6} \mathrm{d}\varphi\ \rho^2\sin\varphi\ \overbrace{\rho^2\sin^2\varphi}^{x^2+y^2} \\ &= \int_0^3 \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi/6} \mathrm{d}\varphi\ \rho^4\sin^3\varphi \end{aligned}$$

(b) In cylindrical coordinates

- $x^2 + y^2 + z^2 \le 9$ is $r^2 + z^2 \le 9$ and
- $\sqrt{3x^2 + 3y^2} \le z$ is $\sqrt{3}\,r \le z$ and
- the integand $x^2 + y^2 = r^2$, and
- $\mathrm{d}V$ is $r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$

Observe that $r^2 + z^2 = 9$ and $\sqrt{3}\,r = z$ intersect when $r^2 + 3r^2 = 9$ so that $r = \frac{3}{2}$ and $z = \frac{3\sqrt{3}}{2}$. Here is a sketch of the $y = 0$ cross–section of T.



So

$$T = \left\{ \ (r\cos\theta\,,\ r\sin\theta\,,\ z) \ \middle|\ 0 \le r \le \tfrac{3}{2},\ 0 \le \theta \le 2\pi,\ \sqrt{3}\,r \le z \le \sqrt{9-r^2} \ \right\}$$

and

$$I = \iiint_V (x^2+y^2)\ \mathrm{d}V = \int_0^{3/2} \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \int_{\sqrt{3}\,r}^{\sqrt{9-r^2}} \mathrm{d}z\ r\ \overbrace{(r^2)}^{x^2+y^2} = \int_0^{3/2} \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \int_{\sqrt{3}\,r}^{\sqrt{9-r^2}} \mathrm{d}z\ \ r^3$$

(c) We'll use the spherical coordinate form.

$$\begin{aligned}
I &= \int_0^3 \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi/6} \mathrm{d}\varphi\ \rho^4 \sin^3\varphi \\
&= \int_0^3 \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \int_0^{\pi/6} \mathrm{d}\varphi\ \rho^4 \sin\varphi \big[1-\cos^2\varphi\big] \\
&= 2\pi \int_0^3 \mathrm{d}\rho\ \rho^4 \left[-\cos\varphi + \frac{\cos^3\varphi}{3}\right]_0^{\pi/6} = 2\pi \left[-\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{8} + 1 - \frac{1}{3}\right] \int_0^3 \mathrm{d}\rho\ \rho^4 \\
&= 2\pi \frac{3^5}{5} \left[\frac{2}{3} - \frac{3\sqrt{3}}{8}\right] = \overbrace{81}^{3^4}\,\pi \left[\frac{4}{5} - \frac{9\sqrt{3}}{20}\right] \approx 5.24
\end{aligned}$$