MATHEMATICS 200 December 2008 Final Exam Solutions

1. A surface is given by
$$z = x^2 - 2xy + y^2.$$

   (a) Find the equation of the tangent plane to the surface at $x = a$, $y = 2a$.

   (b) For what value of a is the tangent plane parallel to the plane $x - y + z = 1$?

   Solution. (a) The surface is $G(x, y, z) = z - x^2 + 2xy - y^2 = 0$. When $x = a$ and $y = 2a$ and (x, y, z) is on the surface, we have $z = a^2 - 2(a)(2a) + (2a)^2 = a^2$. So a normal vector to this surface at $(a, 2a, a^2)$ is

$$\boldsymbol{\nabla} G(a, 2a, a^2) = \langle -2x + 2y\,,\ 2x - 2y\,,\ 1\rangle \Big|_{(x,y,z)=(a,2a,a^2)} = \langle 2a\,,\ -2a\,,\ 1\rangle$$

   and the equation of the tangent plane is

$$\langle 2a\,,\ -2a\,,\ 1\rangle \cdot \big\langle x - a\,,\ y - 2a\,,\ z - a^2\big\rangle = 0 \qquad \text{or} \qquad 2ax - 2ay + z = -a^2$$

   (b) The two planes are parallel when their two normal vectors, namely $\langle 2a\,,\ -2a\,,\ 1\rangle$ and $\langle 1\,,\ -1\,,\ 1\rangle$, are parallel. This is the case if and only if $a = \frac{1}{2}$.

2. The pressure in a solid is given by
$$P(s, r) = sr(4s^2 - r^2 - 2)$$

   where s is the specific heat and r is the density. We expect to measure (s, r) to be approximately $(2, 2)$ and would like to have the most accurate value for P. There are two different ways to measure s and r. Method 1 has an error in s of ± 0.01 and an error in r of ± 0.1, while method 2 has an error of ± 0.02 for both s and r.

   Should we use method 1 or method 2? Explain your reasoning carefully.

   Solution. The linear approximation to $P(s, r)$ at $(2, 2)$ is

$$P(s, r) \approx P(2, 2) + P_s(2, 2)\,(s - 2) + P_r(2, 2)\,(r - 2)$$

   As

$$\begin{aligned} P(2,2) &= (2)(2)\big[4(2)^2 - (2)^2 - 2\big] = 40 \qquad \text{(which we don't actually need)} \\ P_s(2,2) &= \Big[12s^2r - r^3 - 2r\Big]_{s=r=2} = 84 \\ P_r(2,2) &= \Big[4s^3 - 3sr^2 - 2s\Big]_{s=r=2} = 4 \end{aligned}$$

   the linear approximation is

$$P(s, r) \approx 40 + 84\,(s - 2) + 4\,(r - 2)$$

Under method 1, the maximum error in P will have magnitude at most (approximately)

$$84(0.01) + 4(0.1) = 1.24$$

Under method 2, the maximum error in P will have magnitude at most (approximately)

$$84(0.02) + 4(0.02) = 1.76$$

Method 1 is better.

3. $u(x, y)$ is defined as

$$u(x, y) = e^y \, F\big(xe^{-y^2}\big)$$

for an arbitrary function $F(z)$.

(a) If $F(z) = \ln(z)$, find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

(b) For an arbitrary $F(z)$ show that $u(x, y)$ satisfies

$$2xy\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u$$

Solution. For any (differentiable) function F, we have, by the chain rule,

$$\begin{aligned}
\frac{\partial u}{\partial x}(x, y) &= e^y \, F'\big(xe^{-y^2}\big) \; e^{-y^2} \\
\frac{\partial u}{\partial y}(x, y) &= e^y \, F\big(xe^{-y^2}\big) + e^y \, F'\big(xe^{-y^2}\big) \; (-2xy)e^{-y^2}
\end{aligned}$$

(a) In particular, when $F(z) = \ln(z)$, $F'(z) = \frac{1}{z}$ and

$$\begin{aligned}
\frac{\partial u}{\partial x}(x, y) &= e^y \, \frac{1}{xe^{-y^2}} \; e^{-y^2} = \frac{e^y}{x} \\
\frac{\partial u}{\partial y}(x, y) &= e^y \, \ln\big(xe^{-y^2}\big) + e^y \, \frac{1}{xe^{-y^2}} \; (-2xy)e^{-y^2} = e^y \, \ln\big(xe^{-y^2}\big) - 2ye^y \\
&= e^y \, \ln(x) - y^2 \, e^y - 2ye^y
\end{aligned}$$

(b) In general

$$\begin{aligned}
2xy\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} &= 2xy \; e^y \, F'\big(xe^{-y^2}\big) \; e^{-y^2} + e^y \, F\big(xe^{-y^2}\big) + e^y \, F'\big(xe^{-y^2}\big) \; (-2xy)e^{-y^2} \\
&= e^y \, F\big(xe^{-y^2}\big) \\
&= u
\end{aligned}$$

4. The air temperature $T(x, y, z)$ at a location (x, y, z) is given by:

$$T(x, y, z) = 1 + x^2 + yz.$$

(a) A bird passes through $(2,1,3)$ travelling towards $(4,3,4)$ with speed 2. At what rate does the air temperature it experiences change at this instant?

(b) If instead the bird maintains constant altitude ($z=3$) as it passes through $(2,1,3)$ while also keeping at a fixed air temperature, $T=8$, what are its two possible directions of travel?

Solution. The temperature gradient at $(2,1,3)$ is

$$\boldsymbol{\nabla} T(2,1,3) = \langle 2x\,,\,z\,,\,y\rangle\Big|_{(x,y,z)=(2,1,3)} = \langle 4\,,\,3\,,\,1\rangle$$

(a) The bird is flying in the direction $\langle 4-2\,,\,3-1\,,\,4-3\rangle = \langle 2\,,\,2\,,\,1\rangle$ at speed 2 and so has velocity $\mathbf{v} = 2\frac{\langle 2\,,\,2\,,\,1\rangle}{|\langle 2\,,\,2\,,\,1\rangle|} = \frac{2}{3}\langle 2\,,\,2\,,\,1\rangle$. The rate of change of air temperature experienced by the bird at that instant is

$$\boldsymbol{\nabla} T(2,1,3)\cdot\mathbf{v} = \frac{2}{3}\langle 4\,,\,3\,,\,1\rangle\cdot\langle 2\,,\,2\,,\,1\rangle = 10$$

(b) To maintain constant altitude (while not being stationary), the bird's direction of travel has to be of the form $\langle a\,,\,b\,,\,0\rangle$, for some constants a and b, not both zero. To keep the air temperature fixed, its direction of travel has to be perpendicular to $\boldsymbol{\nabla} T(2,1,3) = \langle 4\,,\,3\,,\,1\rangle$. So a and b have to obey

$$0 = \langle a\,,\,b\,,\,0\rangle\cdot\langle 4\,,\,3\,,\,1\rangle = 4a+3b \iff b = -\frac{4}{3}a$$

and the direction of travel has to be a nonzero constant times $\langle 3\,,-4\,,0\rangle$. The two such unit vectors are $\pm\frac{1}{5}\langle 3\,,-4\,,0\rangle$.

5. (a) Find all saddle points, local minima and local maxima of the function

$$f(x,y) = x^3 + x^2 - 2xy + y^2 - x.$$

(b) Use Lagrange multipliers to find the points on the sphere $z^2+x^2+y^2-2y-10=0$ closest to and farthest from the point $(1,-2,1)$.

Solution. (a) To find the critical points we will need the gradient of f, and to apply the second derivative test of Theorem 2.9.16 in the CLP–III text we will need all second order partial derivatives. So we need all partial derivatives of f up to order two. Here they are.

$$\begin{aligned}
f &= x^3 + x^2 - 2xy + y^2 - x \\
f_x &= 3x^2 + 2x - 2y - 1 \qquad & f_{xx} &= 6x+2 \qquad & f_{xy} &= -2 \\
f_y &= -2x + 2y & f_{yy} &= 2 & f_{yx} &= -2
\end{aligned}$$

(Of course, f_{xy} and f_{yx} have to be the same. It is still useful to compute both, as a way to catch some mechanical errors.)

The critical points are the solutions of

$$\begin{aligned} f_x &= 3x^2 + 2x - 2y - 1 = 0 && \text{(E1)} \\ f_y &= -2x + 2y \qquad\qquad = 0 && \text{(E2)} \end{aligned}$$

Substituting $y = x$, from (E2), into (E1) gives

$$3x^2 - 1 = 0 \iff x = \pm\frac{1}{\sqrt{3}} = 0$$

So there are two critical points: $\pm\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

The classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$(2\sqrt{3}+2) \times (2) - (-2)^2 > 0$	$2\sqrt{3}+2 > 0$	local min
$-\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	$(-2\sqrt{3}+2) \times (2) - (-2)^2 < 0$		saddle point

(b) The function $f(x, y, z) = (x-1)^2 + (y+2)^2 + (z-1)^2$ gives the square of the distance from the point (x, y, z) to the point $(1, -2, 1)$. So it suffices to find the (x, y, z) which minimizes $f(x, y, z) = (x-1)^2 + (y+2)^2 + (z-1)^2$ subject to the constraint $g(x, y, z) = z^2 + x^2 + y^2 - 2y - 10 = 0$. By Theorem 2.10.2 in the CLP–III text, any local minimum or maximum (x, y, z) must obey the Lagrange multiplier equations

$$\begin{aligned} f_x &= 2(x-1) = 2\lambda x = \lambda g_x && \text{(E1)} \\ f_y &= 2(y+2) = 2\lambda(y-1) = \lambda g_y && \text{(E2)} \\ f_z &= 2(z-1) = 2\lambda z = \lambda g_z && \text{(E3)} \\ z^2 + x^2 + y^2 - 2y &= 10 && \text{(E4)} \end{aligned}$$

for some real number λ. Now

$$\begin{aligned} \text{(E1)} &\implies x = \frac{1}{1-\lambda} \\ \text{(E2)} &\implies y = -\frac{2+\lambda}{1-\lambda} \\ \text{(E3)} &\implies z = \frac{1}{1-\lambda} \end{aligned}$$

(Note that λ cannot be 1, because if it were (E1) would reduce to $-2 = 0$.) Substituting these into (E4), and using that

$$y - 2 = -\frac{2+\lambda}{1-\lambda} - \frac{2-2\lambda}{1-\lambda} = -\frac{4-\lambda}{1-\lambda}$$

gives

$$\begin{aligned}
&\frac{1}{(1-\lambda)^2} + \frac{1}{(1-\lambda)^2} + \frac{2+\lambda}{1-\lambda}\,\frac{4-\lambda}{1-\lambda} = 10 \\
\iff\ & 2 + (2+\lambda)(4-\lambda) = 10(1-\lambda)^2 \\
\iff\ & 11\lambda^2 - 22\lambda = 0 \\
\iff\ & \lambda = 0 \text{ or } \lambda = 2
\end{aligned}$$

When $\lambda = 0$, we have $(x, y, z) = (1, -2, 1)$ (nasty!), which gives distance zero and so is certainly the closest point. When $\lambda = 2$, we have $(x, y, z) = (-1, 4, -1)$, which does not give distance zero and so is certainly the farthest point.

6. Consider the integral

$$I = \int_0^1 \int_{\sqrt{y}}^1 \frac{\sin(\pi x^2)}{x}\,\mathrm{d}x\,\mathrm{d}y$$

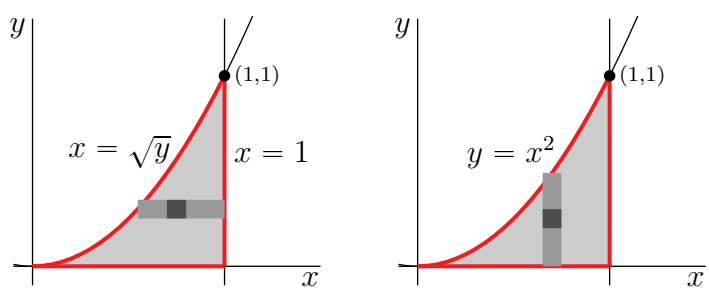
(a) Sketch the region of integration.

(b) Evaluate I.

Solution. (a) On the domain of integration

- y runs from 0 to 1 and
- for each fixed y in that range, x runs from $\sqrt{y}$ to 1.

The figure on the left below is a sketch of that domain, together with a generic horizontal strip as was used in setting up the integral.



(b) The inside integral, $\int_{\sqrt{y}}^1 \frac{\sin(\pi x^2)}{x}\,\mathrm{d}x$, in the given form of I looks really nasty. So let's try exchanging the order of integration. Looking at the figure on the right above, we see that, on the domain of integration,

- x runs from 0 to 1 and
- for each fixed x in that range, y runs from 0 to x^2.

So

$$
\begin{aligned}
I &= \int_0^1 \mathrm{d}x \int_0^{x^2} \mathrm{d}y \; \frac{\sin(\pi x^2)}{x} \\
&= \int_0^1 \mathrm{d}x \; x \sin(\pi x^2) \\
&= \left[-\frac{\cos(\pi x^2)}{2\pi} \right]_0^1 \qquad \text{(Looks pretty rigged!)} \\
&= \frac{1}{\pi}
\end{aligned}
$$

7. Let R be the region bounded on the left by $x = 1$ and on the right by $x^2 + y^2 = 4$. The density in R is

$$\rho(x, y) = \frac{1}{\sqrt{x^2 + y^2}}$$

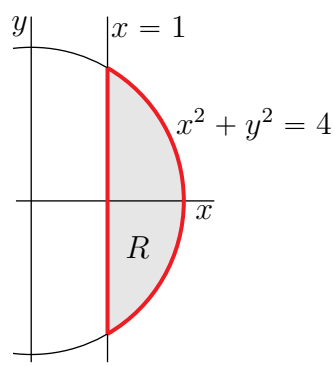
(a) Sketch the region R.

(b) Find the mass of R.

(c) Find the centre-of-mass of R.

Note: You may use the result $\int \sec(\theta) \, \mathrm{d}\theta = \ln|\sec\theta + \tan\theta| + C$.

Solution. (a) Here is a sketch of R.



(b) Considering that

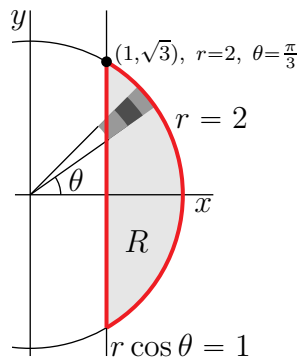
- $\rho(x, y)$ is invariant under rotations about the origin and
- the outer curve $x^2 + y^2 = 4$ is invariant under rotations about the origin and
- the given hint involves a θ integral

we'll use polar coordinates.

Observe that the line $x = 1$ and the circle $x^2 + y^2 = 4$ intersect when

$$1 + y^2 = 4 \iff y = \pm\sqrt{3}$$

and that the polar coordinates of the point $(x, y) = \left(1, \sqrt{3}\right)$ are $r = \sqrt{x^2 + y^2} = 2$ and $\theta = \arctan \frac{y}{x} = \arctan \sqrt{3} = \frac{\pi}{3}$. Looking at the sketch



we see that, on R,

- θ runs from $-\frac{\pi}{3}$ to $\frac{\pi}{3}$ and
- for each fixed θ in that range, r runs from $\frac{1}{\cos\theta} = \sec\theta$ to 2.
- In polar coordinates, $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$, and
- the density $\rho = \frac{1}{\sqrt{x^2+y^2}} = \frac{1}{r}$

So the mass is

$$\begin{aligned}
M &= \iint_R \rho(x, y)\ \mathrm{d}A = \int_{-\pi/3}^{\pi/3} \mathrm{d}\theta \int_{\sec\theta}^{2} \mathrm{d}r\ \frac{r}{r} = \int_{-\pi/3}^{\pi/3} \mathrm{d}\theta\ \left[2 - \sec\theta\right] \\
&= 2\int_0^{\pi/3} \mathrm{d}\theta\ \left[2 - \sec\theta\right] \\
&= 2\Big[2\theta - \ln\left(\sec\theta + \tan\theta\right)\Big]_0^{\pi/3} \\
&= 2\left[\frac{2\pi}{3} - \ln\left(2 + \sqrt{3}\right) + \ln\left(1 + 0\right)\right] \\
&= \frac{4\pi}{3} - 2\ln\left(2 + \sqrt{3}\right)
\end{aligned}$$

(c) By definition, the centre of mass is $(\bar{x}, \bar{y})$, with $\bar{x}$ and $\bar{y}$ being the weighted averages of the x and y–coordinates, respectively, over R. That is,

$$\bar{x} = \frac{\iint_R x\ \rho(x, y)\ \mathrm{d}A}{\iint_R \rho(x, y)\ \mathrm{d}A} \qquad \bar{y} = \frac{\iint_R y\ \rho(x, y)\ \mathrm{d}A}{\iint_R \rho(x, y)\ \mathrm{d}A}$$

By symmetry under reflection in the x–axis, we have $\bar{y} = 0$. So we just have to determine $\bar{x}$. The numerator is

$$\begin{aligned}
\iint_R x\ \rho(x,y)\ \mathrm{d}A &= \int_{-\pi/3}^{\pi/3} \mathrm{d}\theta \int_{\sec\theta}^{2} \mathrm{d}r\ \frac{r}{r}\ \overbrace{r\cos\theta}^{x} \\
&= \frac{1}{2}\int_{-\pi/3}^{\pi/3} \mathrm{d}\theta\ \left[4 - \sec^2\theta\right]\cos\theta = \int_0^{\pi/3} \mathrm{d}\theta\ \left[4\cos\theta - \sec\theta\right] \\
&= \Big[4\sin\theta - \ln\big(\sec\theta + \tan\theta\big)\Big]_0^{\pi/3} \\
&= \left[4\frac{\sqrt{3}}{2} - \ln\left(2+\sqrt{3}\right) + \ln\left(1+0\right)\right] \\
&= 2\sqrt{3} - \ln\left(2+\sqrt{3}\right)
\end{aligned}$$

All together, $\bar{y} = 0$ and

$$\bar{x} = \frac{2\sqrt{3} - \ln\left(2+\sqrt{3}\right)}{\frac{4\pi}{3} - 2\ln\left(2+\sqrt{3}\right)} \approx 1.38$$

8. Let

$$I = \iiint_T xz\ \mathrm{d}V$$

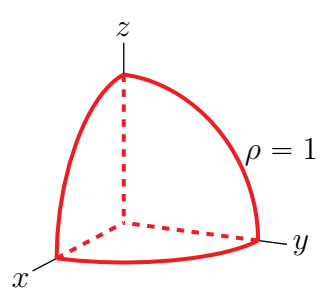
where T is the eighth of the sphere $x^2 + y^2 + z^2 \le 1$ with $x, y, z \ge 0$.

(a) Sketch the volume T.

(b) Express I as a triple integral in spherical coordinates.

(c) Evaluate I by any method.

Solution. (a) Here is a sketch



(b) On T,

- the spherical coordinate φ runs from 0 (the positive z–xis) to $\pi/2$ (the xy–plane), and

- for each fixed φ in that range, θ runs from 0 to $\pi/2$, and
- for each fixed φ and θ, the spherical coordinate ρ runs from 0 to 1.
- In spherical coordinates $\mathrm{d}V = \rho^2 \sin\varphi \,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi$ and

$$xz = \big(\rho\sin\varphi\cos\theta\big)\big(\rho\cos\varphi\big) = \rho^2 \sin\varphi \;\cos\varphi\;\cos\theta$$

So

$$I = \int_0^{\pi/2} \mathrm{d}\varphi \int_0^{\pi/2} \mathrm{d}\theta \int_0^1 \mathrm{d}\rho\; \rho^4 \sin^2\varphi\cos\varphi\;\cos\theta$$

(c) In spherical coordinates,

$$\begin{aligned} I &= \left[\int_0^{\pi/2} \mathrm{d}\varphi\;\sin^2\varphi\cos\varphi\right]\left[\int_0^{\pi/2} \mathrm{d}\theta\;\cos\theta\right]\left[\int_0^1 \mathrm{d}\rho\;\rho^4\right] \\ &= \left[\frac{\sin^3\varphi}{3}\right]_0^{\pi/2} \big[\sin\theta\big]_0^{\pi/2} \left[\frac{\rho^5}{5}\right]_0^1 \\ &= \frac{1}{15} \end{aligned}$$