MATHEMATICS 200 April 2007 Final Exam Solutions

1. A plane Π passes through the points $A = (1, 1, 3)$, $B = (2, 0, 2)$ and $C = (2, 1, 0)$ in $\mathbb{R}^3$.

   (a) Find an equation for the plane Π.

   (b) Find the point E in the plane Π such that the line L through $D = (6, 1, 2)$ and E is perpendicular to Π.

   Solution. (a) The vector from C to A, namely $\langle 1-2\,,\,1-1\,,\,3-0\rangle = \langle -1\,,\,0\,,\,3\rangle$ lies entirely inside Π. The vector from C to B, namely $\langle 2-2\,,\,0-1\,,\,2-0\rangle = \langle 0\,,\,-1\,,\,2\rangle$ also lies entirely inside Π. Consequently, the vector

$$\langle -1\,,\,0\,,\,3\rangle \times \langle 0\,,\,-1\,,\,2\rangle = \det \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\mathbf{k}} \\ -1 & 0 & 3 \\ 0 & -1 & 2 \end{bmatrix} = \langle 3\,,\,2\,,\,1\rangle$$

   is perpendicular to Π. The equation of Π is then

$$\langle 3\,,\,2\,,\,1\rangle \cdot \langle x-2\,,\,y-1\,,\,z\rangle = 0 \qquad \text{or} \qquad 3x + 2y + z = 8$$

   (b) Let E be (x, y, z). Then the vector from D to E, namely $\langle x-6\,,\,y-1\,,\,z-2\rangle$ has to be parallel to the vector $\langle 3\,,\,2\,,\,1\rangle$, which is perpendicular to Π. That is, there must be a number t such that

$$\langle x-6\,,\,y-1\,,\,z-2\rangle = t\,\langle 3\,,\,2\,,\,1\rangle$$
$$\text{or } x = 6+3t,\ \ y = 1+2t,\ \ z = 2+t$$

   As (x, y, z) must be in Π,

$$8 = 3x + 2y + z = 3(6+3t) + 2(1+2t) + (2+t) = 22 + 14t \implies t = -1$$

   So $(x, y, z) = \big(6+3(-1)\,,\,1+2(-1)\,,\,2+(-1)\big) = \big(3\,,\,-1\,,\,1\big)$.

2. Consider the function f that maps each point (x, y) in $\mathbb{R}^2$ to ye^{-x}.

   (a) Suppose that $x = 1$ and $y = e$, but errors of size 0.1 are made in measuring each of x and y. Estimate the maximum error that this could cause in $f(x, y)$.

   (b) The graph of the function f sits in $\mathbb{R}^3$, and the point $(1, e, 1)$ lies on that graph. Find a nonzero vector that is perpendicular to that graph at that point.

   Solution. We are going to need the first order partial derivatives of $f(x, y) = ye^{-x}$ at $(x, y) = (1, e)$. Here they are.

$$f_x(x,y) = -ye^{-x} \qquad\qquad f_x(1,e) = -e\,e^{-1} = -1$$
$$f_y(x,y) = e^{-x} \qquad\qquad f_y(1,e) = e^{-1}$$

(a) The linear approximation to $f(x,y)$ at $(x,y)=(1,e)$ is

$$f(x,y) \approx f(1,e) + f_x(1,e)\,(x-1) + f_y(1,e)\,(y-e) = 1-(x-1)+e^{-1}(y-e)$$

The maximum error is then approximately

$$-1(-0.1)+e^{-1}(0.1) = \frac{1+e^{-1}}{10}$$

(b) The equation of the graph is $g(x,y,z)=f(x,y)-z=0$. Any vector that is a nonzero constant times

$$\boldsymbol{\nabla} g(1,e,1) = \langle f_x(1,e)\,,\, f_y(1,e)\,,\, -1\rangle = \left\langle -1\,,\, e^{-1}\,,\, -1\right\rangle$$

is perpendicular to $g=0$ at $(1,e,1)$.

3. A mosquito is at the location $(3,2,1)$ in $\mathbb{R}^3$. She knows that the temperature T near there is given by $T=2x^2+y^2-z^2$.

   (a) She wishes to stay at the same temperature, but must fly in some initial direction. Find a direction in which the initial rate of change of the temperature is 0.

   (b) If you and another student both get correct answers in part (a), must the directions you give be the same? Why or why not?

   (c) What initial direction or directions would suit the mosquito if she wanted to cool down as fast as possible?

Solution. (a) The temperature gradient at $(3,2,1)$ is

$$\boldsymbol{\nabla} T(3,2,1) = \langle 4x\,,\, 2y\,,\, -2z\rangle\Big|_{(x,y,z)=(3,2,1)} = \langle 12\,,\, 4\,,\, -2\rangle$$

She wishes to fly in a direction that is perpendicular to $\boldsymbol{\nabla} T(3,2,1)$. That is, she wishes to fly in a direction $\langle a\,,\, b\,,\, c\rangle$ that obeys

$$0 = \langle 12\,,\, 4\,,\, -2\rangle \cdot \langle a\,,\, b\,,\, c\rangle = 12a+4b-2c$$

Any nonzero $\langle a\,,\, b\,,\, c\rangle$ that obeys $12a+4b-2c=0$ is an allowed direction. Four allowed unit vectors are $\pm\frac{\langle 0\,,1\,,2\rangle}{\sqrt{5}}$ and $\pm\frac{\langle 1\,,-3\,,0\rangle}{\sqrt{10}}$.

(b) No they need not be the same. Four different explicit directions were given in part (a).

(c) To cool down as quickly as possible, she should move in the direction opposite to the temperature gradient. A unit vector in that direction is $-\frac{\langle 6\,,2\,,-1\rangle}{\sqrt{41}}$.

4. Let F be a function on $\mathbb{R}^2$. Denote points in $\mathbb{R}^2$ by (u, v) and the corresponding partial derivatives of F by $F_u(u, v)$, $F_v(u, v)$, $F_{uu}(u, v)$, $F_{uv}(u, v)$, etc.. Assume those derivatives are all continuous. Express

$$\frac{\partial^2}{\partial x\, \partial y} F(x^2 - y^2, 2xy)$$

in terms of partial derivatives of the function F.

Hint: Let $u = x^2 - y^2$, and $v = 2xy$.

Solution. By the chain rule

$$\begin{aligned}
\frac{\partial}{\partial y} F(x^2 - y^2, 2xy) &= F_u(x^2 - y^2, 2xy)\ (-2y) + F_v(x^2 - y^2, 2xy)\,(2x) \\
\frac{\partial^2}{\partial x\, \partial y} F(x^2 - y^2, 2xy) &= \frac{\partial}{\partial x} \left\{ -2yF_u(x^2 - y^2, 2xy) + 2xF_v(x^2 - y^2, 2xy) \right\} \\
&= -4xyF_{uu}(x^2 - y^2, 2xy) - 4y^2 F_{uv}(x^2 - y^2, 2xy) \\
&\qquad + 2F_v(x^2 - y^2, 2xy) + 4x^2 F_{vu}(x^2 - y^2, 2xy) \\
&\qquad + 4xyF_{vv}(x^2 - y^2, 2xy) \\
&= 2\, F_v(x^2 - y^2, 2xy) - 4xy\, F_{uu}(x^2 - y^2, 2xy) \\
&\qquad + 4(x^2 - y^2)\, F_{uv}(x^2 - y^2, 2xy) \\
&\qquad + 4xy\, F_{vv}(x^2 - y^2, 2xy)
\end{aligned}$$

5. Find all critical points for $f(x, y) = x(x^2 + xy + y^2 - 9)$. Also find out which of these points give local maximum values for $f(x, y)$, which give local minimum values, and which give saddle points.

Solution. To find the critical points we will need the gradient of f, and to apply the second derivative test of Theorem 2.9.16 in the CLP–III text we will need all second order partial derivatives. So we need all partial derivatives of f up to order two. Here they are.

$$\begin{aligned}
f &= x^3 + x^2y + xy^2 - 9x \\
f_x &= 3x^2 + 2xy + y^2 - 9 \qquad & f_{xx} &= 6x + 2y \qquad & f_{xy} &= 2x + 2y \\
f_y &= x^2 + 2xy & f_{yy} &= 2x & f_{yx} &= 2x + 2y
\end{aligned}$$

(Of course, f_{xy} and f_{yx} have to be the same. It is still useful to compute both, as a way to catch some mechanical errors.)

The critical points are the solutions of

$$f_x = 3x^2 + 2xy + y^2 - 9 = 0 \tag{E1}$$
$$f_y = x(x + 2y) \qquad = 0 \tag{E2}$$

Equation (E2) is satisfied if at least one of $x = 0$, $x = -2y$.

- If $x = 0$, equation (E1) reduces to $y^2 - 9 = 0$, which is satisfied if $y = \pm 3$.
- If $x = -2y$, equation (E1) reduces to

$$0 = 3(-2y)^2 + 2(-2y)y + y^2 - 9 = 9y^2 - 9$$

  which is satisfied if $y = \pm 1$.

So there are four critical points: $(0, 3)$, $(0, -3)$, $(-2, 1)$ and $(2, -1)$. The classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0, 3)$	$(6) \times (0) - (6)^2 < 0$		saddle point
$(0, -3)$	$(-6) \times (0) - (-6)^2 < 0$		saddle point
$(-2, 1)$	$(-10) \times (-4) - (-2)^2 > 0$	-10	local max
$(2, -1)$	$(10) \times (4) - (2)^2 > 0$	10	local min

6. Find the largest and smallest values of x^2y^2z in the part of the plane $2x+y+z = 5$ where $x \geq 0$, $y \geq 0$ and $z \geq 0$. Also find all points where those extreme values occur.

Solution. The region of interest is

$$D = \left\{ \ (x, y, z) \ \middle| \ x \geq 0, \ y \geq 0, \ z \geq 0, \ 2x + y + z = 5 \ \right\}$$

First observe that, on the boundary of this region, at least one of x, y and z is zero. So $f(x, y, z) = x^2y^2z$ is zero on the boundary. As f takes values which are strictly bigger than zero at all points of D that are not on the boundary, the minimum value of f is 0 on

$$\partial D = \left\{ \ (x, y, z) \ \middle| \ x \geq 0, \ y \geq 0, \ z \geq 0, \ 2x + y + z = 5, \ \text{at least one of } x, y, z \text{ zero} \ \right\}$$

The maximum value of f will be taken at a critical point. On D

$$f = x^2y^2(5 - 2x - y) = 5x^2y^2 - 2x^3y^2 - x^2y^3$$

So the critical points are the solutions of

$$\begin{aligned} 0 &= f_x(x, y) = 10xy^2 - 6x^2y^2 - 2xy^3 \\ 0 &= f_y(x, y) = 10x^2y - 4x^3y - 3x^2y^2 \end{aligned}$$

or, dividing by the first equation by xy^2 and the second equation by x^2y, (recall that $x, y \neq 0$)

$$\begin{aligned} 10 - 6x - 2y &= 0 \qquad \text{or} \qquad 3x + y = 5 \\ 10 - 4x - 3y &= 0 \qquad \text{or} \qquad 4x + 3y = 10 \end{aligned}$$

Substituting $y = 5 - 3x$, from the first equation, into the second equation gives

$$4x + 3(5 - 3x) = 10 \implies -5x + 15 = 10 \implies x = 1, \ y = 5 - 3(1) = 2$$

So the maximum value of f is $(1)^2(2)^2(5 - 2 - 2) = 4$ at $(1, 2, 1)$.

7. A region E in the xy–plane has the property that for all continuous functions f

$$\iint_E f(x,y)\,\mathrm{d}A = \int_{x=-1}^{x=3}\left[\int_{y=x^2}^{y=2x+3} f(x,y)\mathrm{d}y\right]\mathrm{d}x$$

(a) Compute $\iint_E x\,\mathrm{d}A$.

(b) Sketch the region E.

(c) Set up $\iint_E x\,\mathrm{d}A$ as an integral or sum of integrals in the opposite order.

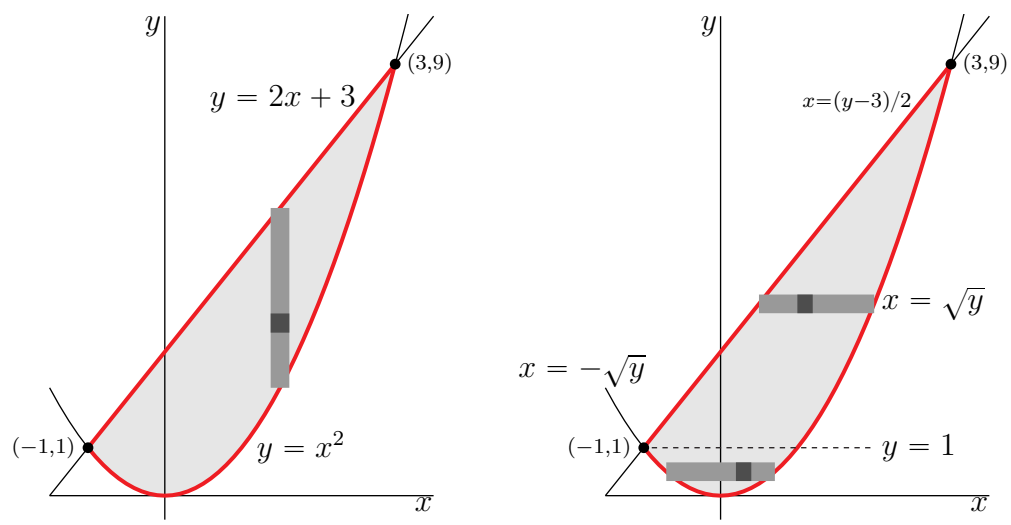
Solution. (a) When $f(x,y) = x$,

$$\begin{aligned}\int_{x=-1}^{x=3}\left[\int_{y=x^2}^{y=2x+3} x\mathrm{d}y\right]\mathrm{d}x &= \int_{x=-1}^{x=3}\left[x(2x+3-x^2)\right]\mathrm{d}x \\ &= \left[\frac{2x^3}{3}+\frac{3x^2}{2}-\frac{x^4}{4}\right]_{-1}^{3} = 18+\frac{27}{2}-\frac{81}{4}+\frac{2}{3}-\frac{3}{2}+\frac{1}{4} \\ &= 18+12-20+\frac{2}{3} = \frac{32}{3}\end{aligned}$$

(b) On the region E

- x runs from -1 to 3 and
- for each x in that range, y runs from x^2 to $2x+3$

Here are two sketches of E, with the left one including a generic vertical strip as was used in setting up the given integral.



(c) To reverse the order of integration we use horizontal strips as in the figure on the right above. Looking at that figure, we see that, on the region E,

- y runs from 0 to 9 and
- for each y between 0 and 1, x runs from $-\sqrt{y}$ to $\sqrt{y}$
- for each y between 1 and 9, x runs from $(y-3)/2$ to $\sqrt{y}$

So

$$I = \int_0^1 \mathrm{d}y \int_{-\sqrt{y}}^{\sqrt{y}} \mathrm{d}x\ x + \int_1^9 \mathrm{d}y \int_{(y-3)/2}^{\sqrt{y}} \mathrm{d}x\ x$$

8. A certain solid V is a right–circular cylinder. Its base is the disk of radius 2 centred at the origin in the xy–plane. It has height 2 and density $\sqrt{x^2+y^2}$.

   A smaller solid U is obtained by removing the inverted cone, whose base is the top surface of V and whose vertex is the point $(0,0,0)$.

   (a) Use cylindrical coordinates to set up an integral giving the mass of U.

   (b) Use spherical coordinates to set up an integral giving the mass of U.

   (c) Find that mass.

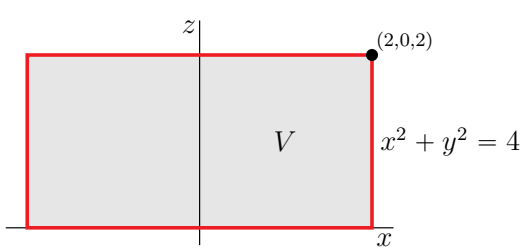
Solution. The disk of radius 2 centred at the origin in the xy–plane is $x^2+y^2 \le 4$. So

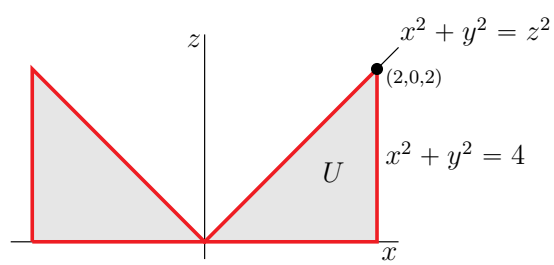
$$V = \left\{\ (x,y,z)\ \middle|\ x^2+y^2 \le 4,\ 0 \le z \le 2\ \right\}$$

The cone with vertex at the origin that contains the top edge, $x+y^2=4$, $z=2$, of U is $x^2+y^2=z^2$. So

$$U = \left\{\ (x,y,z)\ \middle|\ x^2+y^2 \le 4,\ 0 \le z \le 2,\ x^2+y^2 \ge z^2\ \right\}$$

Here are sketches of the $y=0$ cross–section of V, on the left, and U, on the right.





(a) In cylindrical coordinates, $x^2+y^2 \le 4$ becomes $r \le 2$ and $x^2+y^2 \ge z^2$ is $r \ge |z|$, and the density is $\sqrt{x^2+y^2} = r$. So

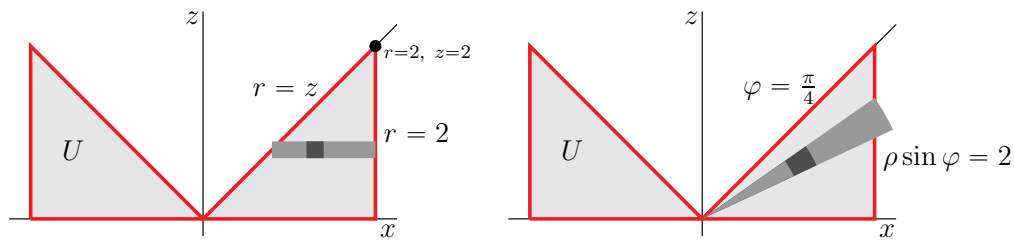
$$U = \left\{\ (r\cos\theta, r\sin\theta, z)\ \middle|\ r \le 2,\ 0 \le z \le 2,\ r \ge z\ \right\}$$

Looking at the figure on the left below, we see that, on U

- z runs from 0 to 2, and
- for each z is that range, r runs from z to 2 and θ runs from 0 to 2π.
- $\mathrm{d}V = r\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$

So

$$\text{Mass} = \int_0^2 \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta \int_z^2 \mathrm{d}r\ r\ \overbrace{r}^{\text{density}} = \int_0^2 \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta \int_z^2 \mathrm{d}r\ r^2$$



(b) Recall that in spherical coordinates,

$$\begin{aligned} x &= \rho \sin\varphi \cos\theta \\ y &= \rho \sin\varphi \sin\theta \\ z &= \rho \cos\varphi \\ x^2 + y^2 &= \rho^2 \sin^2\varphi \end{aligned}$$

so that $x^2 + y^2 \le 4$ becomes $\rho \sin\varphi \le 2$, and $x^2 + y^2 \ge z^2$ becomes

$$\rho \sin\varphi \ge \rho\cos\varphi \iff \tan\varphi \ge 1 \iff \varphi \ge \frac{\pi}{4}$$

and the density $\sqrt{x^2+y^2} = \rho\sin\varphi$. So

$$U = \left\{\ (\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\ \middle|\ \pi/4 \le \varphi \le \pi/2,\ 0 \le \theta \le 2\pi,\ \rho\sin\varphi \le 2\ \right\}$$

Looking at the figure on the right above, we see that, on U

- φ runs from $\frac{\pi}{4}$ (on the cone) to $\frac{\pi}{2}$ (on the xy–plane), and
- for each φ is that range, ρ runs from 0 to $\frac{2}{\sin\varphi}$ and θ runs from 0 to 2π.
- $\mathrm{d}V = \rho^2 \sin\varphi\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\varphi$

So

$$\begin{aligned} \text{Mass} &= \int_{\pi/4}^{\pi/2} \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\theta \int_0^{2/\sin\varphi} \mathrm{d}\rho\ \rho^2 \sin\varphi\ \overbrace{\rho\sin\varphi}^{\text{density}} \\ &= \int_{\pi/4}^{\pi/2} \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\theta \int_0^{2/\sin\varphi} \mathrm{d}\rho\ \rho^3 \sin^2\varphi \end{aligned}$$

(c) We'll use the cylindrical form.

$$\begin{aligned}
\text{Mass} &= \int_0^2 \mathrm{d}z \int_0^{2\pi} \mathrm{d}\theta \int_z^2 \mathrm{d}r\ r^2 \\
&= 2\pi \int_0^2 \mathrm{d}z\ \frac{8 - z^3}{3} \\
&= \frac{2\pi}{3}\ \left[16 - \frac{2^4}{4}\right] \\
&= 8\pi
\end{aligned}$$