MATHEMATICS 200 December 2006 Final Exam Solutions

1. Consider the surface given by:

$$z^3 - xyz^2 - 4x = 0.$$

(a) Find expressions for $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ as functions of x, y, z.

(b) Evaluate $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ at $(1, 1, 2)$.

(c) Measurements are made with errors, so that $x = 1 \pm 0.03$ and $y = 1 \pm 0.02$. Find the corresponding maximum error in measuring z.

(d) A particle moves over the surface along the path whose projection in the xy–plane is given in terms of the angle θ as

$$x(\theta) = 1 + \cos\theta, \;\; y(\theta) = \sin\theta$$

from the point $A : x = 2, \; y = 0$ to the point $B : x = 1, \; y = 1$. Find $\frac{\mathrm{d}z}{\mathrm{d}\theta}$ at points A and B.

Solution. (a) We are told that

$$z(x, y)^3 - xy\, z(x, y)^2 - 4x = 0$$

for all (x, y) (sufficiently near $(1, 1)$). Differentiating this equation with respect to x gives

$$\begin{aligned} & 3z(x, y)^2 \frac{\partial z}{\partial x}(x, y) - y\, z(x, y)^2 - 2xy\, z(x, y) \frac{\partial z}{\partial x}(x, y) - 4 = 0 \\ \Longrightarrow \; & \frac{\partial z}{\partial x} = \frac{4 + yz^2}{3z^2 - 2xyz} \end{aligned}$$

and differentiating with respect to y gives

$$\begin{aligned} & 3z(x, y)^2 \frac{\partial z}{\partial y}(x, y) - x\, z(x, y)^2 - 2xy\, z(x, y) \frac{\partial z}{\partial y}(x, y) = 0 \\ \Longrightarrow \; & \frac{\partial z}{\partial y} = \frac{xz^2}{3z^2 - 2xyz} \end{aligned}$$

(b) When $(x, y, z) = (1, 1, 2)$,

$$\frac{\partial z}{\partial x}(1, 1) = \frac{4 + (1)(2)^2}{3(2)^2 - 2(1)(1)(2)} = 1 \qquad \frac{\partial z}{\partial y}(1, 1) = \frac{(1)(2)^2}{3(2)^2 - 2(1)(1)(2)} = \frac{1}{2}$$

(c) Under the linear approximation at $(1, 1)$

$$z(x, y) \approx z(1, 1) + z_x(1, 1)\,(x - 1) + z_y\,(1, 1)\,(y - 1) = 2 + (x - 1) + \frac{1}{2}(y - 1)$$

So errors of ± 0.03 in x and ± 0.02 in y leads of errors of about

$$\pm\left[0.03 + \frac{1}{2}(0.02)\right] = \pm 0.04$$

in z.

(d) By the chain rule

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta} z\big(x(\theta), y(\theta)\big) &= z_x\big(x(\theta), y(\theta)\big)\, x'(\theta) + z_y\big(x(\theta), y(\theta)\big)\, y'(\theta) \\ &= -z_x\big(1+\cos\theta, \sin\theta\big)\, \sin\theta + z_y\big(1+\cos\theta, \sin\theta\big)\, \cos\theta\end{aligned}$$

At A, $x = 2$, $y = 0$, $z = 2$ (since $z^3 - (2)(0)z^2 - 4(2) = 0$) and $\theta = 0$, so that

$$\frac{\partial z}{\partial x}(2,0) = \frac{4 + (0)(2)^2}{3(2)^2 - 2(2)(0)(2)} = \frac{1}{3} \qquad \frac{\partial z}{\partial y}(2,0) = \frac{(2)(2)^2}{3(2)^2 - 2(2)(0)(2)} = \frac{2}{3}$$

and

$$\frac{\mathrm{d}z}{\mathrm{d}\theta} = -\frac{1}{3}\sin(0) + \frac{2}{3}\cos(0) = \frac{2}{3}$$

At B, $x = 1$, $y = 1$, $z = 2$ and $\theta = \frac{\pi}{2}$, so that, by part (b),

$$\frac{\partial z}{\partial x}(1,1) = 1 \qquad \frac{\partial z}{\partial y}(1,1) = \frac{1}{2}$$

and

$$\frac{\mathrm{d}z}{\mathrm{d}\theta} = -\sin\frac{\pi}{2} + \frac{1}{2}\cos\frac{\pi}{2} = -1$$

2. A hiker is walking on a mountain with height above the $z = 0$ plane given by

$$z = f(x,y) = 6 - xy^2$$

The positive x–axis points east and the positive y–axis points north, and the hiker starts from the point $P(2,1,4)$.

(a) In what direction should the hiker proceed from P to ascend along the steepest path? What is the slope of the path?

(b) Walking north from P, will the hiker start to ascend or descend? What is the slope?

(c) In what direction should the hiker walk from P to remain at the same height?

Solution. (a) The gradient of f at $(x, y) = (2, 1)$ is

$$\boldsymbol{\nabla} f(2,1) = \left\langle -y^2\,,\ -2xy \right\rangle \Big|_{(x,y)=(2,1)} = \langle -1\,,\ -4 \rangle$$

So the path of steepest ascent is in the direction $-\frac{1}{\sqrt{17}}\langle 1\,,\ 4\rangle$, which is a little west of south. The slope is

$$|\boldsymbol{\nabla} f(2,1)| = |\,\langle -1\,,\ -4\rangle\,| = \sqrt{17}$$

(b) The directional derivative in the north direction is

$$D_{\langle 0,1\rangle} f(2,1) = \boldsymbol{\nabla} f(2,1) \cdot \langle 0,1\rangle = \langle -1\,,\ -4\rangle \cdot \langle 0,1\rangle = -4$$

So the hiker descends with slope $|-4| = 4$.

(c) To contour, i.e. remain at the same height, the hiker should walk in a direction perpendicular to $\boldsymbol{\nabla} f(2,1) = \langle -1\,,\ -4\rangle$. Two unit vectors perpendicular to $\langle -1\,,\ -4\rangle$ are $\pm\frac{1}{\sqrt{17}}\langle 4, -1\rangle$.

3. (a) Find and classify all critical points of the function

$$f(x,y) = x^3 - y^3 - 2xy + 6.$$

(b) Use the method of Lagrange Multipliers to find the maximum and minimum values of

$$f(x,y) = xy$$

subject to the constraint

$$x^2 + 2y^2 = 1.$$

Solution. (a) To find the critical points we will need the gradient of f, and to apply the second derivative test of Theorem 2.9.16 in the CLP–III text we will need all second order partial derivatives. So we need all partial derivatives of f up to order two. Here they are.

$$\begin{aligned} f &= x^3 - y^3 - 2xy + 6 \\ f_x &= 3x^2 - 2y \qquad & f_{xx} &= 6x \qquad & f_{xy} &= -2 \\ f_y &= -3y^2 - 2x \qquad & f_{yy} &= -6y \qquad & f_{yx} &= -2 \end{aligned}$$

The critical points are the solutions of

$$f_x = 3x^2 - 2y = 0 \qquad f_y = -3y^2 - 2x = 0$$

Substituting $y = \frac{3}{2}x^2$, from the first equation, into the second equation gives

$$\begin{aligned} -3\left(\frac{3}{2}x^2\right)^2 - 2x = 0 &\iff -2x\left(\frac{3^3}{2^3}x^3 + 1\right) = 0 \\ &\iff x = 0,\ -\frac{2}{3} \end{aligned}$$

So there are two critical points: $(0,0)$, $\left(-\frac{2}{3}, \frac{2}{3}\right)$.

The classification is

critical point	$f_{xx}f_{yy} - f_{xy}^2$	f_{xx}	type
$(0,0)$	$0 \times 0 - (-2)^2 < 0$		saddle point
$\left(-\frac{2}{3}, \frac{2}{3}\right)$	$(-4) \times (-4) - (-2)^2 > 0$	-4	local max

(b) For this problem the objective function is $f(x,y) = xy$ and the constraint function is $g(x,y) = x^2 + 2y^2 - 1$. To apply the method of Lagrange multipliers we need $\boldsymbol{\nabla} f$ and $\boldsymbol{\nabla} g$. So we start by computing the first order derivatives of these functions.

$$f_x = y \qquad f_y = x \qquad g_x = 2x \qquad g_y = 4y$$

So, according to the method of Lagrange multipliers, we need to find all solutions to

$$y = \lambda(2x) \tag{E1}$$
$$x = \lambda(4y) \tag{E2}$$
$$x^2 + 2y^2 - 1 = 0 \tag{E3}$$

First observe that none of x, y, λ can be zero, because if at least one of them is zero, then (E1) and (E2) force $x = y = 0$, which violates (E3). Dividing (E1) by (E2) gives $\frac{y}{x} = \frac{x}{2y}$ so that $x^2 = 2y^2$ or $x = \pm\sqrt{2}\,y$. Then (E3) gives

$$2y^2 + 2y^2 = 1 \iff y = \pm\frac{1}{2}$$

The method of Lagrange multipliers, Theorem 2.10.2 in the CLP–III text, gives that the only possible locations of the maximum and minimum of the function f are $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2}\right)$. So the maximum and minimum values of f are $\frac{1}{2\sqrt{2}}$ and $-\frac{1}{2\sqrt{2}}$, respectively.

4. The integral I is defined as

$$I = \iint_R f(x,y)\,\mathrm{d}A = \int_1^{\sqrt{2}} \int_{1/y}^{\sqrt{y}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_{\sqrt{2}}^{4} \int_{y/2}^{\sqrt{y}} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

(a) Sketch the region R.

(b) Re–write the integral I by reversing the order of integration.

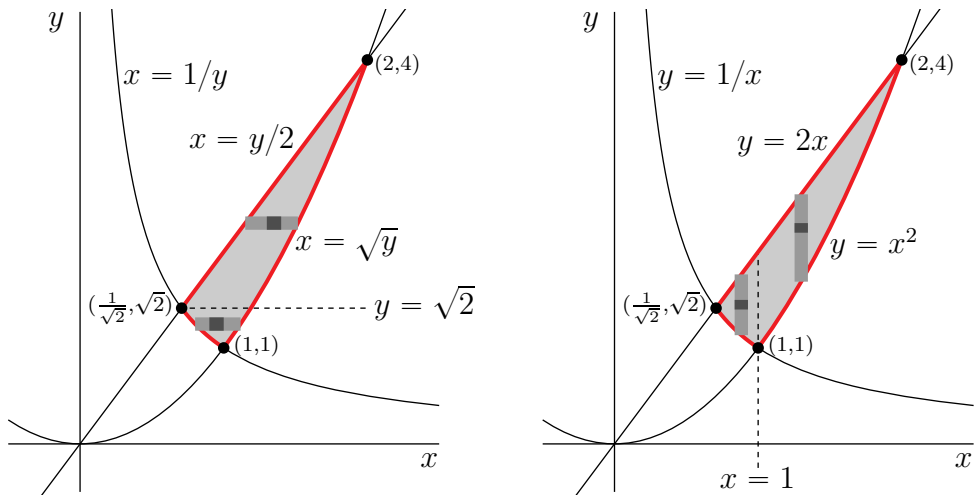
(c) Compute the integral I when $f(x,y) = x/y$.

Solution. (a) On R

- y runs from 1 to 4 (from 1 to $\sqrt{2}$ in the first integral and from $\sqrt{2}$ to 4 in the second).

- For each fixed y between 1 and $\sqrt{2}$, x runs from $\frac{1}{y}$ to $\sqrt{y}$ and
- for each fixed y between $\sqrt{2}$ and 4, x runs from $\frac{y}{2}$ to $\sqrt{y}$.

The figure on the left below is a sketch of R, together with generic horizontal strips as were used in setting up the integral.



(b) To reverse the order of integration we use vertical strips as in the figure on the right above. Looking at that figure, we see that, on R,

- x runs from $1/\sqrt{2}$ to 2.
- For each fixed x between $1/\sqrt{2}$ and 1, y runs from $\frac{1}{x}$ to $2x$ and
- for each fixed x between 1 and 2, y runs from x^2 to $2x$.

So

$$I = \int_{1/\sqrt{2}}^{1} \int_{1/x}^{2x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x + \int_{1}^{2} \int_{x^2}^{2x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

(c) When $f(x,y) = \frac{x}{y}$,

$$\begin{aligned}
I &= \int_{1}^{\sqrt{2}} \int_{1/y}^{\sqrt{y}} \frac{x}{y}\,\mathrm{d}x\,\mathrm{d}y + \int_{\sqrt{2}}^{4} \int_{y/2}^{\sqrt{y}} \frac{x}{y}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_{1}^{\sqrt{2}} \frac{1}{y}\left[\frac{y}{2} - \frac{1}{2y^2}\right]\mathrm{d}y + \int_{\sqrt{2}}^{4} \frac{1}{y}\left[\frac{y}{2} - \frac{y^2}{8}\right]\mathrm{d}y \\
&= \left[\frac{y}{2} + \frac{1}{4y^2}\right]_1^{\sqrt{2}} + \left[\frac{y}{2} - \frac{y^2}{16}\right]_{\sqrt{2}}^{4} = \frac{1}{\sqrt{2}} + \frac{1}{8} - \frac{1}{2} - \frac{1}{4} + 2 - 1 - \frac{1}{\sqrt{2}} + \frac{1}{8} \\
&= \frac{1}{2}
\end{aligned}$$

5. (a) Sketch the region $\mathcal{L}$ (in the first quadrant of the xy–plane) with boundary curves

$$x^2 + y^2 = 2,\ x^2 + y^2 = 4,\ y = x,\ y = 0.$$

The mass of a thin lamina with a density function $\rho(x, y)$ over the region $\mathcal{L}$ is given by

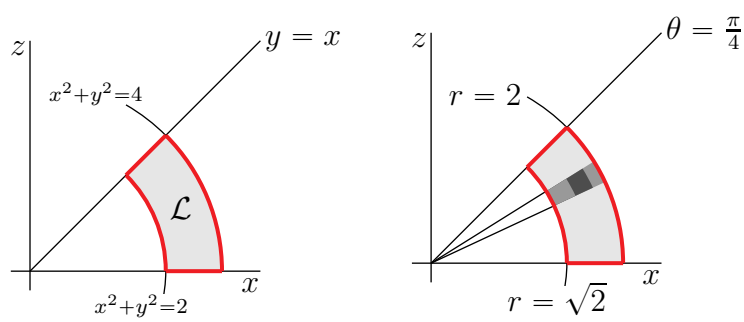
$$M = \iint_{\mathcal{L}} \rho(x, y)\, \mathrm{d}A$$

(b) Find an expression for M as an integral in polar coordinates.

(c) Find M when

$$\rho(x, y) = \frac{2xy}{x^2 + y^2}$$

Solution. (a) The region $\mathcal{L}$ is sketched in the figure on the leflt below.



(b) In polar coordinates

- the circle $x^2 + y^2 = 2$ is $r^2 = 2$ or $r = \sqrt{2}$, and
- the circle $x^2 + y^2 = 4$ is $r^2 = 4$ or $r = 2$, and
- the line $y = x$ is $r \sin\theta = r \cos\theta$, or $\tan\theta = 1$, or (for the part in the first quadrant) $\theta = \frac{\pi}{4}$, and
- the positive x–axis ($y = 0$, $x \geq 0$) is $\theta = 0$

Looking at the figure on the right above, we see that, in $\mathcal{L}$,

- θ runs from 0 to $\frac{\pi}{4}$, and
- for each fixed θ in that range, r runs from $\sqrt{2}$ to 2.
- $\mathrm{d}A$ is $r\, \mathrm{d}r\, \mathrm{d}\theta$

So

$$M = \int_0^{\pi/4} \mathrm{d}\theta \int_{\sqrt{2}}^{2} \mathrm{d}r\ r\, \rho(r\cos\theta\,,\, r\sin\theta)$$

(c) When

$$\rho = \frac{2xy}{x^2+y^2} = \frac{2r^2\cos\theta\,\sin\theta}{r^2} = \sin(2\theta)$$

we have

$$\begin{aligned} M &= \int_0^{\pi/4} \mathrm{d}\theta \int_{\sqrt{2}}^{2} \mathrm{d}r\ r\ \sin(2\theta) \\ &= \left[\int_0^{\pi/4} \sin(2\theta)\ \mathrm{d}\theta\right]\left[\int_{\sqrt{2}}^{2} r\ \mathrm{d}r\right] \\ &= \left[-\frac{1}{2}\cos(2\theta)\right]_0^{\pi/4}\left[\frac{r^2}{2}\right]_{\sqrt{2}}^{2} = \frac{1}{2}\ \frac{4-2}{2} \\ &= \frac{1}{2} \end{aligned}$$

6. (a) A triple integral $\iiint_E f(x,y,z)\ \mathrm{d}V$ is given in the iterated form

$$J = \int_0^1 \int_0^{1-\frac{x}{2}} \int_0^{4-2x-4z} f(x,y,z)\ \mathrm{d}y\,\mathrm{d}z\,\mathrm{d}x$$

(i) Sketch the domain E in 3–dimensions. Be sure to show the units.

(ii) Rewrite the integral as one or more iterated integrals in the form

$$J = \int_{y=}^{y=} \int_{x=}^{x=} \int_{z=}^{z=} f(x,y,z)\ \mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$$

(b) Use spherical coordinates to evaluate the integral

$$I = \iiint_D z\ \mathrm{d}V$$

where D is the solid enclosed by the cone $z = \sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2 = 4$.

Solution. (a) (i) In the given integral J,

- x runs from 0 to 1,
- for each fixed x in that range, z runs from 0 to $1-\frac{x}{2}$, and
- for each fixed x and z as above, y runs from 0 to $4-2x-4z$.

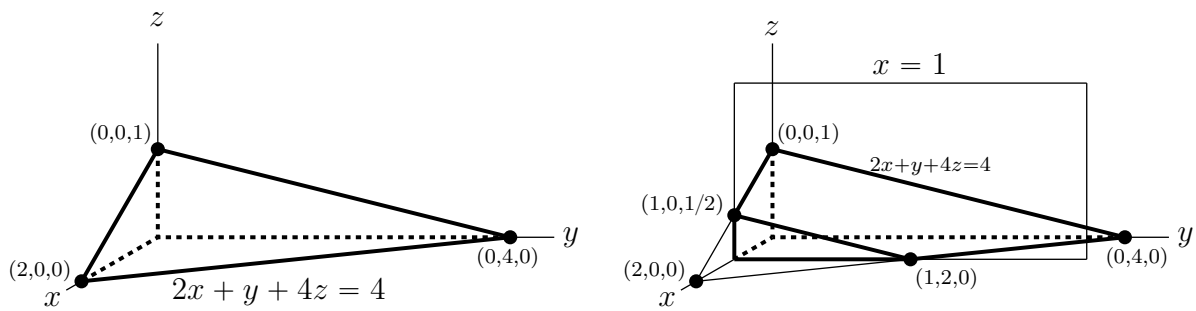
So

$$E = \left\{\ (x,y,z)\ \middle|\ 0 \le x \le 1,\ 0 \le z \le 1-\tfrac{x}{2},\ 0 \le y \le 4-2x-4z\ \right\}$$

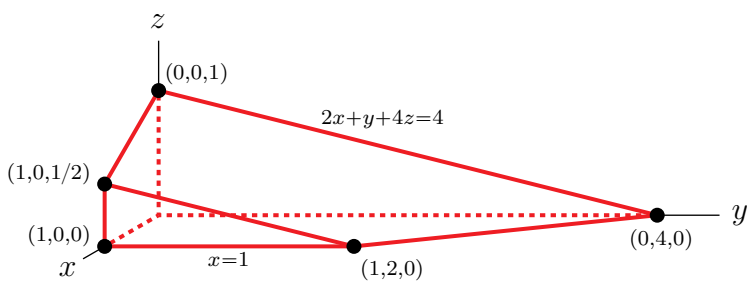
Notice that the condition $y \le 4 - 2x - 4z$ can be rewritten as $z \le 1 - \frac{x}{2} - \frac{y}{4}$. When $y \ge 0$, this implies that $z \le 1 - \frac{x}{2}$, so that we can drop the condition $z \le 1 - \frac{x}{2}$ from our description of E:

$$E = \left\{ \ (x, y, z) \ \middle| \ 0 \le x \le 1, \ 0 \le y \le 4 - 2x - 4z, \ z \ge 0 \ \right\}$$

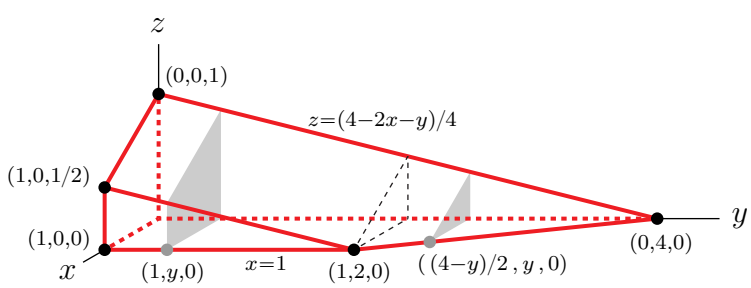
First, we figure out what E looks like. The plane $2x+y+4z = 4$ intersects the x–, y– and z–axes at $(2, 0, 0)$, $(0, 4, 0)$ and $(0, 0, 1)$, respectively. That plane is shown in the sketch on the left below. The set of points $\left\{ \ (x, y, z) \ \middle| \ x, y, z \ge 0, \ y \le 4 - 2x - 4z \ \right\}$ is outlined with heavy lines.



So it only remains to impose the condtion $x \le 1$, which chops off the front bit of the tetrahedron. This is done in the sketch on the right above. Here is a cleaned up sketch of E.



(a) (ii) We are to reorder the integration so that the outside integral is over y, the middle integral is over x, and the inside integral is over z. Looking at the figure below,



we see that

- y runs from 0 to 4, and
- for each fixed y in that range, (x, z) runs over

$$\left\{ (x,z) \;\middle|\; 0 \le x \le 1,\ 2x + 4z \le 4 - y,\ z \ge 0 \right\}$$

- for each fixed y between 0 and 2 (as in the left hand shaded bit in the figure above)
  - x runs from 0 to 1, and then
  - for each fixed x in that range, z runs from 0 to $\frac{4-2x-y}{4}$.
- for each fixed y between 2 and 4 (as in the right hand shaded bit in the figure above)
  - x runs from 0 to $\frac{4-y}{2}$ (the line of intersection of the plane $2x + y + 4z = 4$ and the xy–plane is $z = 0$, $2x + y = 4$), and then
  - for each fixed x in that range, z runs from 0 to $\frac{4-2x-y}{4}$.

So the integral

$$J = \int_{y=0}^{y=2} \int_{x=0}^{x=1} \int_{z=0}^{z=\frac{4-2x-y}{4}} f(x,y,z)\ \mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y + \int_{y=2}^{y=4} \int_{x=0}^{x=\frac{4-y}{4}} \int_{z=0}^{z=\frac{4-2x-y}{4}} f(x,y,z)\ \mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$$

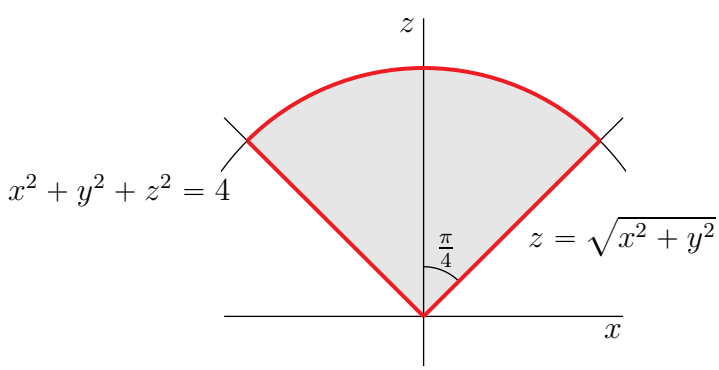
(b) Recall that in spherical coordinates,

$$\begin{aligned} x &= \rho \sin\varphi \cos\theta \\ y &= \rho \sin\varphi \sin\theta \\ z &= \rho \cos\varphi \\ x^2 + y^2 &= \rho^2 \sin^2\varphi \end{aligned}$$

so that $x^2 + y^2 + z^2 = 4$ becomes $\rho = 2$, and $\sqrt{x^2 + y^2} = z$ becomes

$$\rho \sin\varphi = \rho \cos\varphi \iff \tan\varphi = 1 \iff \varphi = \frac{\pi}{4}$$

Here is a sketch of the $y = 0$ cross–section of D.



Looking at the figure above, we see that, on D

- φ runs from 0 (the positive z–axis) to $\frac{\pi}{4}$ (on the cone), and
- for each φ is that range, ρ runs from 0 to 2 and θ runs from 0 to 2π.

So

$$D = \left\{ \ (\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi) \ \middle| \ 0 \le \varphi \le {}^{\pi}/_{4}, \ 0 \le \theta \le 2\pi, \ \rho \le 2 \ \right\}$$

and, as $\mathrm{d}V = \rho^2 \sin\varphi \, \mathrm{d}\rho \, \mathrm{d}\theta \, \mathrm{d}\varphi$,

$$\begin{aligned}
I &= \int_0^{\pi/4} \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\theta \int_0^2 \mathrm{d}\rho \ \rho^2 \sin\varphi \ \overbrace{\rho \cos\varphi}^{z} \\
&= \int_0^{\pi/4} \mathrm{d}\varphi \int_0^{2\pi} \mathrm{d}\theta \int_0^2 \mathrm{d}\rho \ \rho^3 \sin\varphi \ \cos\varphi \\
&= 2\pi \ \frac{2^4}{4} \int_0^{\pi/4} \mathrm{d}\varphi \ \ \sin\varphi \ \cos\varphi \\
&= 2\pi \ \frac{2^4}{4} \left[\frac{\sin^2\varphi}{2} \right]_0^{\pi/4} \\
&= 2\pi
\end{aligned}$$