MATHEMATICS 200 April 2006 Final Exam Solutions

1. If two resistors of resistance R_1 and R_2 are wired in parallel, then the resulting resistance R satisfies the equation $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. Use the linear approximation to estimate the change in R if R_1 decreases from 2 to 1.9 ohms and R_2 increases from 8 to 8.1 ohms.

   Solution. The function $R(R_1, R_2)$ is defined implictly by

$$\frac{1}{R(R_1, R_2)} = \frac{1}{R_1} + \frac{1}{R_2} \qquad (*)$$

   In particular

$$\frac{1}{R(2,8)} = \frac{1}{2} + \frac{1}{8} = \frac{5}{8} \implies R(2,8) = \frac{8}{5}$$

   We wish to use the linear approximation

$$R(R_1, R_2) \approx R(2,8) + \frac{\partial R}{\partial R_1}(2,8)\,(R_1 - 2) + \frac{\partial R}{\partial R_2}(2,8)\,(R_2 - 8)$$

   To do so, we need the partial derivatives $\frac{\partial R}{\partial R_1}(2,8)$ and $\frac{\partial R}{\partial R_2}(2,8)$. To find them, we differentiate $(*)$ with respect to R_1 and R_2:

$$-\frac{1}{R(R_1, R_2)^2}\frac{\partial R}{\partial R_1}(R_1, R_2) = -\frac{1}{R_1^2}$$
$$-\frac{1}{R(R_1, R_2)^2}\frac{\partial R}{\partial R_2}(R_1, R_2) = -\frac{1}{R_2^2}$$

   Setting $R_1 = 2$ and $R_2 = 8$ gives

$$-\frac{1}{(8/5)^2}\frac{\partial R}{\partial R_1}(2,8) = -\frac{1}{4} \quad \implies \quad \frac{\partial R}{\partial R_1}(2,8) = \frac{16}{25}$$
$$-\frac{1}{(8/5)^2}\frac{\partial R}{\partial R_2}(2,8) = -\frac{1}{64} \quad \implies \quad \frac{\partial R}{\partial R_2}(2,8) = \frac{1}{25}$$

   So the specified change in R is

$$R(1.9, 8.1) - R(2,8) \approx \frac{16}{25}(-0.1) + \frac{1}{25}(0.1) = -\frac{15}{250} = -0.06$$

2. Assume that the directional derivative of $w = f(x, y, z)$ at a point P is a maximum in the direction of the vector $2\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}$, and the value of the directional derivative in that direction is $3\sqrt{6}$.

   (a) Find the gradient vector of $w = f(x, y, z)$ at P.

(b) Find the directional derivative of $w = f(x, y, z)$ at P in the direction of the vector $\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$

Solution. (a) Use $\boldsymbol{\nabla} f(P)$ to denote the gradient vector of f at P. We are told that

- directional derivative of f at P is a maximum in the direction $2\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}$, which implies that $\boldsymbol{\nabla} f(P)$ is parallel to $2\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}$, and
- the magnitude of the directional derivative in that direction is $3\sqrt{6}$, which implies that $|\boldsymbol{\nabla} f(P)| = 3\sqrt{6}$.

So

$$\boldsymbol{\nabla} f(P) = 3\sqrt{6}\frac{2\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}}{|2\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + \hat{\mathbf{k}}|} = 6\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 3\hat{\mathbf{k}}$$

(b) The directional derivative of f at P in the direction $\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$ is

$$\boldsymbol{\nabla} f(P) \cdot \frac{\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}}{|\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}|} = \frac{1}{\sqrt{2}}\big(6\hat{\boldsymbol{\imath}} - 3\hat{\boldsymbol{\jmath}} + 3\hat{\mathbf{k}}\big) \cdot \big(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}\big) = \frac{3}{\sqrt{2}}$$

3. Use the Second Derivative Test to find all values of the constant c for which the function $z = x^2 + cxy + y^2$ has a saddle point at $(0, 0)$.

Solution. Write $f(x, y) = x^2 + cxy + y^2$. Then

$$\begin{aligned} f_x(x, y) &= 2x + cy & \qquad f_x(0, 0) &= 0 \\ f_y(x, y) &= cx + 2y & \qquad f_y(0, 0) &= 0 \\ f_{xx}(x, y) &= 2 \\ f_{xy}(x, y) &= c \\ f_{yy}(x, y) &= 2 \end{aligned}$$

As $f_x(0, 0) = f_y(0, 0) = 0$, we have that $(0, 0)$ is always a critical point for f. According to the Second Derivative Test, $(0, 0)$ is also a saddle point for f if

$$f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}(0, 0)^2 < 0 \iff 4 - c^2 < 0 \iff |c| > 2$$

As a remark, the Second Derivative Test provides no information when the expression $f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}(0, 0)^2 = 0$, i.e. when $c = \pm 2$. But when $c = \pm 2$,

$$f(x, y) = x^2 \pm 2xy + y^2 = (x \pm y)^2$$

and f has a local minimum, not a saddle point, at $(0, 0)$.

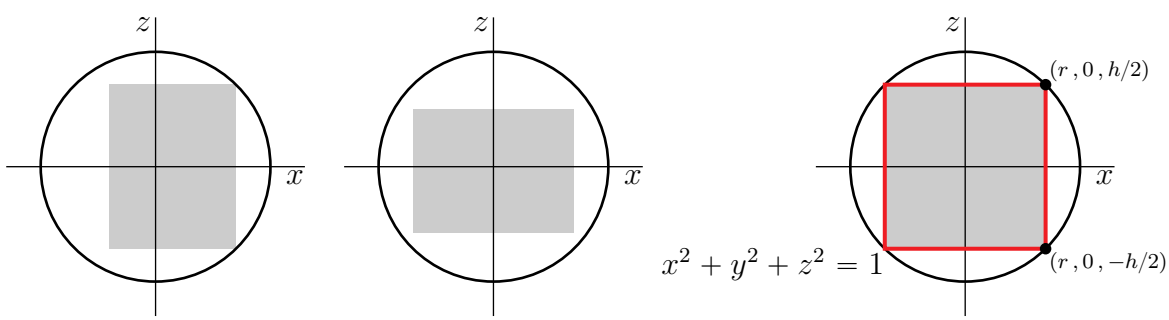
4. Use the Method of Lagrange Multipliers to find the radius of the base and the height of a right circular cylinder of maximum volume which can be fit inside the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution. Let r and h denote the radius and height, respectively, of the cylinder. We can always choose our coordinate system so that the axis of the cylinder is parallel to the z–axis.

- If the axis of the cylinder does not lie exactly on the z–axis, we can enlarge the cylinder sideways. (See the figure on the left below. It shows the $y = 0$ cross–section of the cylinder.) So we can assume that the axis of the cylinder lies on the z–axis
- If the top and/or the bottom of the cylinder does not touch the sphere $x^2+y^2+z^2 = 1$, we can enlarge the cylinder vertically. (See the central figure below.)
- So we may assume that the cylinder is

$$\left\{ (x, y, z) \;\middle|\; x^2 + y^2 \le r^2,\ -h/2 \le z \le h/2 \right\}$$

with $r^2 + (h/2)^2 = 1$. See the figure on the right below.



So we are to maximize the volume, $f(r, h) = \pi r^2 h$, of the cylinder subject to the constraint $g(r, h) = r^2 + \frac{h^2}{4} - 1 = 0$. According to the method of Lagrange multipliers, we need to find all solutions to

$$f_r = 2\pi r h = 2\lambda r = \lambda g_r \tag{E1}$$

$$f_h = \pi r^2 = \lambda \frac{h}{2} = \lambda g_h \tag{E2}$$

$$r^2 + \frac{h^2}{4} = 1 \tag{E3}$$

Equation (E1), $2r(\pi h - \lambda) = 0$, gives that either $r = 0$ or $\lambda = \pi h$. Clearly $r = 0$ cannot give the maximum volume, so $\lambda = \pi h$. Substituting $\lambda = \pi h$ into (E2) gives

$$\pi r^2 = \frac{1}{2}\pi h^2 \implies r^2 = \frac{h^2}{2}$$

Substituting $r^2 = \frac{h^2}{2}$ into (E3) gives

$$\frac{h^2}{2} + \frac{h^2}{4} = 1 \implies h^2 = \frac{4}{3}$$

Clearly both r and h have to be positive, so $h = \frac{2}{\sqrt{3}}$ and $r = \sqrt{\frac{2}{3}}$.

5. Let $z = f(x, y)$ where $x = 2s + t$ and $y = s - t$. Find the values of the constants a, b and c such that

$$a\frac{\partial^2 z}{\partial x^2} + b\frac{\partial^2 z}{\partial x\,\partial y} + c\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial s^2} + \frac{\partial^2 z}{\partial t^2}$$

You may assume that $z = f(x, y)$ is a smooth function so that the Chain Rule and Clairaut's Theorem on the equality of the mixed partial derivatives apply.

Solution. The notation in the statement of this question is horrendous — the symbol z is used with two different meanings in one equation. On the left hand side, it is a function of x and y, and on the right hand side, it is a function of s and t. Unfortunately that abuse of notation is also very common. Let us undo the notation conflict by renaming the function of s and t to $F(s, t)$. That is

$$F(s, t) = f\big(2s + t\,,\ s - t\big)$$

In this new notation, we are being asked to find a, b and c so that

$$a\frac{\partial^2 f}{\partial x^2} + b\frac{\partial^2 f}{\partial x \partial y} + c\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 F}{\partial s^2} + \frac{\partial^2 F}{\partial t^2}$$

with the arguments on the right hand side being (s, t) and the arguments on the left hand side being $\big(2s + t\,,\ s - t\big)$.

By the chain rule,

$$\begin{aligned}
\frac{\partial F}{\partial s}(s, t) &= 2\frac{\partial f}{\partial x}\big(2s + t\,,\ s - t\big) + \frac{\partial f}{\partial y}\big(2s + t\,,\ s - t\big) \\
\frac{\partial^2 F}{\partial s^2}(s, t) &= \frac{\partial}{\partial s}\left[2\frac{\partial f}{\partial x}\big(2s + t\,,\ s - t\big) + \frac{\partial f}{\partial y}\big(2s + t\,,\ s - t\big)\right] \\
&= 4\frac{\partial^2 f}{\partial x^2}\big(2s + t\,,\ s - t\big) + 2\frac{\partial^2 f}{\partial y \partial x}\big(2s + t\,,\ s - t\big) \\
&\quad + 2\frac{\partial^2 f}{\partial x \partial y}\big(2s + t\,,\ s - t\big) + \frac{\partial^2 f}{\partial y^2}\big(2s + t\,,\ s - t\big) \\
\frac{\partial F}{\partial t}(s, t) &= \frac{\partial f}{\partial x}\big(2s + t\,,\ s - t\big) - \frac{\partial f}{\partial y}\big(2s + t\,,\ s - t\big)
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 F}{\partial t^2}(s,t) &= \frac{\partial}{\partial t}\left[\frac{\partial f}{\partial x}(2s+t\,,\,s-t) - \frac{\partial f}{\partial y}(2s+t\,,\,s-t)\right] \\
&= \frac{\partial^2 f}{\partial x^2}(2s+t\,,\,s-t) - \frac{\partial^2 f}{\partial y \partial x}(2s+t\,,\,s-t) \\
&\quad - \frac{\partial^2 f}{\partial x \partial y}(2s+t\,,\,s-t) + \frac{\partial^2 f}{\partial y^2}(2s+t\,,\,s-t)
\end{aligned}
$$

Suppressing the arguments

$$
\frac{\partial^2 F}{\partial s^2} + \frac{\partial^2 F}{\partial t^2} = 5\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x \partial y} + 2\frac{\partial^2 f}{\partial y^2}
$$

Finally, translating back into the notation of the question

$$
\frac{\partial^2 z}{\partial s^2} + \frac{\partial^2 z}{\partial t^2} = 5\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2}
$$

so that $a = 5$ and $b = c = 2$.

6. Combine the sum of the two interated double integrals

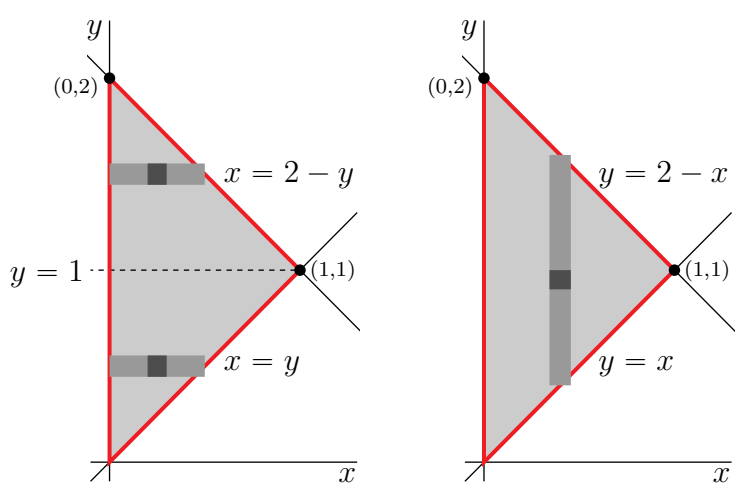
$$
\int_{y=0}^{y=1}\int_{x=0}^{x=y} f(x,y)\ \mathrm{d}x\,\mathrm{d}y + \int_{y=1}^{y=2}\int_{x=0}^{x=2-y} f(x,y)\ \mathrm{d}x\,\mathrm{d}y
$$

into a single interated double integral with the order of integration reversed.

Solution. In the given integrals

- y runs for 0 to 2, and
- for each fixed y between 0 and 1, x runs from 0 to y and
- for each fixed y between 1 and 2, x runs from 0 to $2-y$

The figure on the left below contains a sketch of that region together with the generic horizontal slices that were used to set up the given integrals.



To reverse the order of integration, we switch to vertical, rather than horizontal, slices, as in the figure on the right above. Looking at that figure, we see that

- x runs for 0 to 1, and
- for each fixed x in that range, y runs from x to $2-x$.

So the desired integral is

$$\int_{x=0}^{x=1} \int_{y=x}^{y=2-x} f(x,y)\ \mathrm{d}y\,\mathrm{d}x$$

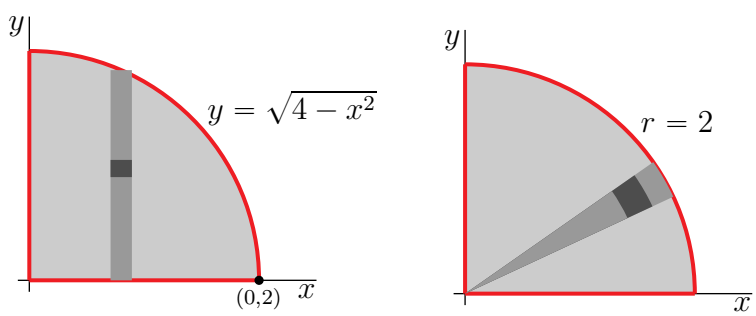
7. Evaluate the iterated double integral

$$\int_{x=0}^{x=2} \int_{y=0}^{y=\sqrt{4-x^2}} (x^2+y^2)^{\frac{3}{2}}\ \mathrm{d}y\,\mathrm{d}x$$

Solution. On the domain of integration

- x runs for 0 to 2, and
- for each fixed x in that range, y runs from 0 to $\sqrt{4-x^2}$. The equation $y=\sqrt{4-x^2}$ is equivalent to $x^2+y^2=4$, $y \geq 0$.

This domain is sketched in the figure on the left below.



Considering that

- the integrand, $(x^2+y^2)^{\frac{3}{2}}$, is invariant under rotations about the origin and
- the outer curve, $x^2+y^2=4$, is invariant under rotations about the origin

we'll use polar coordinates. In polar coordinates,

- the outer curve, $x^2+y^2=4$, is $r=2$, and
- the integrand, $(x^2+y^2)^{\frac{3}{2}}$ is r^3, and
- $\mathrm{d}A = r\,\mathrm{d}r\,\mathrm{d}\theta$

Looking at the figure on the right above, we see that the given integral is, in polar coordinates,

$$\int_0^{\pi/2} \mathrm{d}\theta \int_0^2 \mathrm{d}r\ r(r^3) = \frac{\pi}{2}\,\frac{2^5}{5} = \frac{16\pi}{5}$$

8. Consider the region E in 3–dimensions specified by the spherical inequalities

$$1 \le \rho \le 1 + \cos\varphi$$

(a) Draw a reasonably accurate picture of E in 3–dimensions. Be sure to show the units on the coordinates axes.

(b) Find the volume of E.

Solution. (a) First observe that both boundaries of E, namely $\rho = 1$ and $\rho = 1 + \cos\varphi$, are independent of the spherical coordinate θ. So E is invariant under rotations about the z–axis. To sketch E we

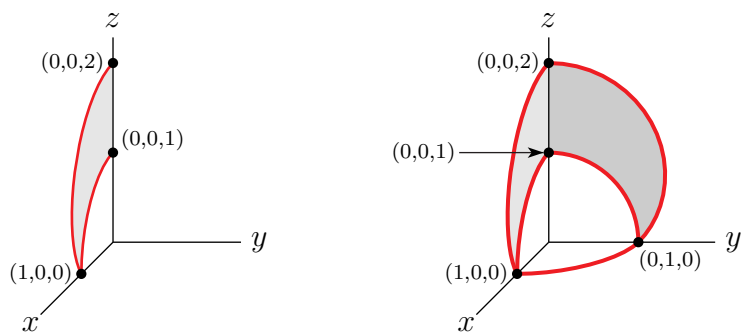
- first sketch the part of the boundary of E with $\theta = 0$ (i.e. in the half of the xz–plane with $x > 0$), and then
- rotate about the z–axis.

The part of the boundary of E with $\theta = 0$ (i.e. in the half–plane $y = 0$, $x \ge 0$), consists of two curves.

- $\rho = 1 + \cos\varphi$, $\theta = 0$:
  - When $\varphi = 0$ (i.e. on the positive z–axis), We have $\cos\varphi = 1$ and hence $\rho = 2$. So this curve starts at $(0, 0, 2)$.
  - As φ increases $\cos\varphi$, and hence ρ, decreases.
  - When φ is $\frac{\pi}{2}$ (i.e. in the xy–plane), we have $\cos\varphi = 0$ and hence $\rho = 1$.
  - When $\frac{\pi}{2} < \varphi \le \pi$, we have $\cos\varphi < 0$ and hence $\rho < 1$. All points in E are required to obey $\rho \ge 1$. So this part of the boundary stops at the point $(1, 0, 0)$ in the xy–plane.
  - The curve $\rho = 1 + \cos\varphi$, $\theta = 0$, $0 \le \varphi \le \frac{\pi}{2}$ is sketched in the figure on the left below. It is the outer curve from $(0, 0, 2)$ to $(1, 0, 0)$.
- $\rho = 1$, $\theta = 0$:
  - The surface $\rho = 1$ is the sphere of radius 1 centred on the origin.
  - As we observed above, the conditions $1 \le \rho \le 1 + \cos\varphi$ force $0 \le \varphi \le \frac{\pi}{2}$, i.e. $z \ge 0$.
  - The sphere $\rho = 1$ intersects the quarter plane $y = 0$, $x \ge 0$, $z \ge 0$, in the quarter circle centred on the origin that starts at $(0, 0, 1)$ on the z–axis and ends at $(1, 0, 0)$ in the xy–plane.

– The curve $\rho = 1$, $\theta = 0$, $0 \le \varphi \le \frac{\pi}{2}$ is sketched in the figure on the left below. It is the inner curve from $(0, 0, 1)$ to $(1, 0, 0)$.

To get E, rotate the shaded region in the figure on the left below about the z-axis. The part of E in the first octant is sketched in the figure on the right below. The part of E in the xz–plane (with $x \ge 0$) is lightly shaded and the part of E in the yz–plane (with $y \ge 0$) is shaded a little more darkly.



(b) In E

- φ runs from 0 (i.e. the positive z–axis) to $\frac{\pi}{2}$ (i.e. the xy–plane).
- For each φ in that range ρ runs from 1 to $1 + \cos\varphi$ and θ runs from 0 to 2π.
- In spherical coordinates $\mathrm{d}V = \rho^2 \sin\varphi \, \mathrm{d}\rho \, \mathrm{d}\theta \, \mathrm{d}\varphi$.

So

$$\begin{aligned}
\text{Volume}(E) &= \int_0^{\pi/2} \mathrm{d}\varphi \int_1^{1+\cos\varphi} \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\theta \; \rho^2 \sin\varphi \\
&= 2\pi \int_0^{\pi/2} \mathrm{d}\varphi \;\; \sin\varphi \; \frac{(1+\cos\varphi)^3 - 1^3}{3} \\
&= -\frac{2\pi}{3} \int_2^1 \left(u^3 - 1\right) \mathrm{d}u \qquad \text{with } u = 1 + \cos\varphi, \;\; \mathrm{d}u = -\sin\varphi \, \mathrm{d}\varphi \\
&= -\frac{2\pi}{3} \left[\frac{u^4}{4} - u\right]_2^1 \\
&= -\frac{2\pi}{3} \left[\frac{1}{4} - 1 - 4 + 2\right] \\
&= \frac{11\pi}{6}
\end{aligned}$$