MATHEMATICS 200 December 2013 Final Exam

1. Short Answer Problems. Show your work. Not all questions are of equal difficulty. Simplify your answers as much as possible in this question.

   (a) The line L has vector parametric equation $\mathbf{r}(t) = (2+3t)\hat{\boldsymbol{\imath}} + 4t\hat{\boldsymbol{\jmath}} - \hat{\mathbf{k}}$.

   i. Write the symmetric equations for L.

   ii. Let α be the angle between the line L and the plane given by the equation $x - y + 2z = 0$. Find α.

   (b) i. Find the equation of the tangent plane to the surface $x^2z^3 + y\sin(\pi x) = -y^2$ at the point $P = (1, 1, -1)$.

   ii. Let z be defined implicitly by $x^2z^3 + y\sin(\pi x) = -y^2$. Find $\frac{\partial z}{\partial x}$ at the point $P = (1, 1, -1)$.

   iii. Let z be the same implicit function as in part (ii), defined by the equation $x^2z^3 + y\sin(\pi x) = -y^2$. Let $x = 0.97$, and $y = 1$. Find the approximate value of z.

   (c) Suppose that $u = x^2 + yz$, $x = \rho r\cos(\theta)$, $y = \rho r\sin(\theta)$ and $z = \rho r$. Find $\frac{\partial u}{\partial r}$ at the point $(\rho_0, r_0, \theta_0) = (2, 3, \pi/2)$.

   (d) Let $f(x)$ be a differentiable function, and suppose it is given that $f'(0) = 10$. Let $g(s,t) = f(as - bt)$, where a and b are constants. Evaluate $\frac{\partial g}{\partial s}$ at the point $(s,t) = (b,a)$, that is, find $\left.\frac{\partial g}{\partial s}\right|_{(b,a)}$.

   (e) Suppose it is known that the direction of the fastest increase of the function $f(x,y)$ at the origin is given by the vector $\langle 1, 2\rangle$. Find a unit vector u that is tangent to the level curve of $f(x,y)$ that passes through the origin.

   (f) Find all the points on the surface $x^2 + 9y^2 + 4z^2 = 17$ where the tangent plane is parallel to the plane $x - 8z = 0$.

   (g) Find the total mass of the rectangular box $[0,1] \times [0,2] \times [0,3]$ (that is, the box defined by the inequalities $0 \le x \le 1$, $0 \le y \le 2$, $0 \le z \le 3$), with density function $h(x,y,z) = x$.

2. The shape of a hill is given by $z = 1000 - 0.02x^2 - 0.01y^2$. Assume that the x–axis is pointing East, and the y–axis is pointing North, and all distances are in metres.

   (a) What is the direction of the steepest ascent at the point $(0, 100, 900)$? (The answer should be in terms of directions of the compass).

   (b) What is the slope of the hill at the point $(0, 100, 900)$ in the direction from (a)?

   (c) If you ride a bicycle on this hill in the direction of the steepest descent at 5 m/s, what is the rate of change of your altitude (with respect to time) as you pass through the point (0, 100, 900)?

3. (a) Find the minimum of the function

$$f(x, y, z) = (x-2)^2 + (y-1)^2 + z^2$$

subject to the constraint $x^2 + y^2 + z^2 = 1$, using the method of Lagrange multipliers.

(b) Give a geometric interpretation of this problem.

4. (a) Find the minimum of the function $h(x, y) = -4x - 2y + 6$ on the closed bounded domain defined by $x^2 + y^2 \leq 1$.

(b) Explain why Question 4 gives another way of solving Question 3.

5. This question is about the integral

$$\int_0^1 \int_{\sqrt{3}y}^{\sqrt{4-y^2}} \ln\left(1 + x^2 + y^2\right) \, \mathrm{d}x \, \mathrm{d}y$$

(a) Sketch the domain of integration.

(b) Evaluate the integral by transforming to polar coordinates.

6. Evaluate

$$\int_{-1}^{0} \int_{-2}^{2x} e^{y^2} \, \mathrm{d}y \, \mathrm{d}x$$

7. Let $a > 0$ be a fixed positive real number. Consider the solid inside both the cylinder $x^2 + y^2 = ax$ and the sphere $x^2 + y^2 + z^2 = a^2$. Compute its volume.

Hint: $\int \sin^3(\theta) = \frac{1}{12}\cos(3\theta) - \frac{3}{4}\cos(\theta) + C$

8. (a) Sketch the surface given by the equation $z = 1 - x^2$.

(b) Let E be the solid bounded by the plane $y = 0$, the cylinder $z = 1 - x^2$, and the plane $y = z$. Set up the integral

$$\iiint_E f(x, y, z) \, \mathrm{d}V$$

as an iterated integral.

9. (a) Find the volume of the solid inside the surface defined by the equation $\rho = 8sin(\varphi)$ in spherical coordinates.

Hint: you do not need a sketch to answer this question; and

$$\int \sin^4(\varphi) = \frac{1}{32}\big(12\varphi - 8\sin(2\varphi) + \sin(4\varphi)\big) + C$$

(b) Sketch this solid or describe what it looks like.

Hint: it is a solid of revolution.