MATHEMATICS 200 December 2008 Final Exam

1. A surface is given by
   \[ z = x^2 - 2xy + y^2. \]
   (a) Find the equation of the tangent plane to the surface at \( x = a, \ y = 2a \).
   (b) For what value of \( a \) is the tangent plane parallel to the plane \( x - y + z = 1 \)?

2. The pressure in a solid is given by
   \[ P(s, r) = sr(4s^2 - r^2 - 2) \]
   where \( s \) is the specific heat and \( r \) is the density. We expect to measure \((s, r)\) to be approximately \((2, 2)\) and would like to have the most accurate value for \( P \). There are two different ways to measure \( s \) and \( r \). Method 1 has an error in \( s \) of \( \pm 0.01 \) and an error in \( r \) of \( \pm 0.1 \), while method 2 has an error of \( \pm 0.02 \) for both \( s \) and \( r \).
   Should we use method 1 or method 2? Explain your reasoning carefully.

3. \( u(x, y) \) is defined as
   \[ u(x, y) = e^y F(xe^{-y^2}) \]
   for an arbitrary function \( F(z) \).
   (a) If \( F(z) = \ln(z) \), find \( \frac{\partial u}{\partial x} \) and \( \frac{\partial u}{\partial y} \).
   (b) For an arbitrary \( F(z) \) show that \( u(x, y) \) satisfies
   \[ 2xy \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u \]

4. The air temperature \( T(x, y, z) \) at a location \((x, y, z)\) is given by:
   \[ T(x, y, z) = 1 + x^2 + yz. \]
   (a) A bird passes through \((2, 1, 3)\) travelling towards \((4, 3, 4)\) with speed 2. At what rate does the air temperature it experiences change at this instant?
   (b) If instead the bird maintains constant altitude \((z = 3)\) as it passes through \((2, 1, 3)\) while also keeping at a fixed air temperature, \( T = 8 \), what are its two possible directions of travel?

5. (a) Find all saddle points, local minima and local maxima of the function
   \[ f(x, y) = x^3 + x^2 - 2xy + y^2 - x. \]
   (b) Use Lagrange multipliers to find the points on the sphere \( z^2 + x^2 + y^2 - 2y - 10 = 0 \) closest to and farthest from the point \((1, -2, 1)\).
6. Consider the integral

\[ I = \int_0^1 \int_{\sqrt{y}}^1 \frac{\sin(\pi x^2)}{x} \, dx \, dy \]

(a) Sketch the region of integration.
(b) Evaluate I.

7. Let R be the region bounded on the left by \( x = 1 \) and on the right by \( x^2 + y^2 = 4 \). The density in \( R \) is

\[ \rho(x, y) = \frac{1}{\sqrt{x^2 + y^2}} \]

(a) Sketch the region \( R \).
(b) Find the mass of \( R \).
(c) Find the centre-of-mass of \( R \).

Note: You may use the result \( \int \sec(\theta) \, d\theta = \ln|\sec \theta + \tan \theta| + C \).

8. Let

\[ I = \iiint_T xz \, dV \]

where \( T \) is the eighth of the sphere \( x^2 + y^2 + z^2 \leq 1 \) with \( x, y, z \geq 0 \).

(a) Sketch the volume \( T \).
(b) Express \( I \) as a triple integral in spherical coordinates.
(c) Evaluate \( I \) by any method.