II. Linear Systems of Equations

§II.1 The Definition

We are shortly going to develop a systematic procedure which is guaranteed to find every solution to every system of linear equations. The fact that such a procedure exists makes systems of linear equations very unusual. If you pick a system of equations at random (i.e. not from a course or textbook) the odds are that you won't be able to solve it. Fortunately, it is possible to use linear systems to approximate many real world situations. So linear systems are not only easy, but useful. We start by giving a formal definition of "linear system of equations". Then we develop the systematic procedure, which is called Gaussian elimination. Then we consider applications to loaded cables and to finding straight lines (and other curves) that best fit experimental data.

Definition II.1 A system of linear equations is one which may be written in the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \qquad (1)\\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \qquad (2)\\ \vdots \qquad\qquad\qquad &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \qquad (m) \end{aligned}$$

Here, all of the coefficients a_{ij} and all of the right hand sides b_i are assumed to be known constants. All of the x_i's are assumed to be unknowns, that we are to solve for. Note that every left hand side is a sum of terms of the form *constant* $\times\ x_i^1$.

§II.2 Solving Linear Systems of Equations

We now introduce, by way of several examples, the systematic procedure for solving systems of linear equations.

Example II.2 Here is a system of three equations in three unknowns.

$$\begin{aligned} x_1 + x_2 + x_3 &= 4 \qquad (1)\\ x_1 + 2x_2 + 3x_3 &= 9 \qquad (2)\\ 2x_1 + 3x_2 + x_3 &= 7 \qquad (3) \end{aligned}$$

We can reduce the system down to two equations in two unknowns by using the first equation to solve for x_1 in terms of x_2 and x_3

$$x_1 = 4 - x_2 - x_3 \qquad (1')$$

and substituting this solution into the remaining two equations

$$\begin{aligned} &(2) \qquad (4 - x_2 - x_3) + 2x_2 + 3x_3 = 9 \qquad \Longrightarrow \qquad x_2 + 2x_3 = 5\\ &(3) \qquad 2(4 - x_2 - x_3) + 3x_2 + x_3 = 7 \qquad \Longrightarrow \qquad x_2 - x_3 = -1 \end{aligned}$$

We now have two equations in two unknowns, x_2 and x_3. We can solve the first of these two equations for x_2 in terms of x_3

$$x_2 = 5 - 2x_3 \qquad (2')$$

and substitute the result into the final equation

$$(5 - 2x_3) - x_3 = -1 \qquad \Longrightarrow \qquad -3x_3 = -6$$

to get down to one equation in the one unknown x_3. We can trivially solve it for x_3

$$x_3 = 2$$

and substitute the result into (2') to give x_2

$$x_2 = 5 - 2x_3 = 5 - 2 \times 2 = 1$$

and finally substitute the now known values of x_2 and x_3 into (1') to determine x_1

$$x_1 = 4 - x_2 - x_3 = 4 - 1 - 2 = 1$$

We are going to write down a lot of systems of linear equations, so it pays to set up a streamlined notation right away. The new notation is gotten by writing the system in the standard form given in Definition II.1 and then dropping all the unknowns, x_i, all the $+$ signs and all the $=$ signs. Optionally, you can put the resulting array of numbers in square brackets and draw a vertical line where the $=$ signs used to be.

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

This whole beast is called the augmented matrix of the system. The part to the left of the vertical line that has replaced the equal signs, i.e. the part that contains all of the coefficients a_{rc}, is called the coefficient matrix. Line number r contains equation number r, with all of the unknowns, $+$ signs and $=$ signs dropped. The usual numbering convention is that the first index in a_{rc} gives the row number and the second index gives the column number.

The basic strategy for solving linear systems is the one we used in Example II.2. We start with m equations in n unknowns. We use the first equation to eliminate x_1 from equations (2) through (m), leaving $m-1$ equations in $n-1$ unknowns. And repeat until we run out of either equations or variables. There is a method for eliminating x_1 from equations (2) through (m) that is a bit more efficient than solving equation (1) for x_1 in terms of x_2 through x_n and substituting the result into the remaining equations. We shall apply a sequence of "row operations" on our system of equations. Each row operation has the property that it replaces the original system of equations by another system which has **exactly** the same set of solutions. The allowed row operations are

- replace equation (i) by c(i) where c is any **nonzero** number
- replace equation (i) by (i) +c(j). In words, equation (i) is replaced by equation (i) plus c times equation (j). Here any j other than i is allowed.
- interchange (i) and (j)

Any $(x_1, \cdots, x_n)$ that satisfies equations (i) and (j) also satisfies $c(i)$ and $(i) + c(j)$. So application of a row operation does not result in any loss of solutions. Each of these row operations can be reversed by another row operation and so has does not generate new solutions either. On the other hand, multiplying an equation by zero destroys it forever, so multiplying an equation by zero (even in some disguised way) is not a legitimate row operation.

Example II.3 The augmented matrix for the system of equations

$$\begin{aligned} 2x_1 + \ \ x_2 + 3x_3 &= \ \ 1 \\ 4x_1 + 5x_2 + 7x_3 &= \ \ 7 \\ 2x_1 - 5x_2 + 5x_3 &= -7 \end{aligned}$$

is

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 4 & 5 & 7 & 7 \\ 2 & -5 & 5 & -7 \end{array}\right]$$

Ordinary arithmetic errors are a big problem when you do row operations by hand. There is a technique called "the check column" (that is modeled after the "parity bit" in computer hardware design) which provides a very effective way to catch mechanical errors. To implement a "check column" you tack onto the right hand side of the augmented matrix an additional column. Each entry in this check column is the sum of all the entries in the row of the augmented matrix that is to the left of the check column entry. For example, the top entry in the check column is $2 + 1 + 3 + 1 = 7$.

$$\left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 4 & 5 & 7 & 7 \\ 2 & -5 & 5 & -7 \end{array}\right] \begin{array}{c} 7 \\ 23 \\ -5 \end{array}$$

To use the check column you just perform the same row operations on the check column as you do on the augmented matrix. After each row operation you check that each entry in the check column is still the sum of all the entries in the corresponding row of the augmented matrix.

We now want to eliminate the x_1's from equations (2) and (3). That is, we want to make the first entries in rows 2 and 3 of the augmented matrix zero. We can achieve this by subtracting two times row (1) from row (2) and subtracting row (1) from row (3).

$$\begin{array}{c} (1) \\ (2) - 2(1) \\ (3) - (1) \end{array} \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & 3 & 1 & 5 \\ 0 & -6 & 2 & -8 \end{array}\right] \begin{array}{c} 7 \\ 9 \\ -12 \end{array}$$

Observe that the check column entry 9 is the sum $0 + 3 + 1 + 5$ of the entries in the second row of the augmented matrix. If this were not the case, it would mean that we made a mechanical error. Similarly the check column entry -12 is the sum $0 - 6 + 2 - 8$.

We have now succeeded in eliminating all of the x_1's from equations (2) and (3). For example, row 2 now stands for the equation

$$3x_2 + x_3 = 5$$

We next use equation (2) to eliminate all x_2's from equation (3).

$$\begin{array}{c} (1) \\ (2) \\ (3) + 2(2) \end{array} \left[\begin{array}{ccc|c} 2 & 1 & 3 & 1 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & 4 & 2 \end{array}\right] \begin{array}{c} 7 \\ 9 \\ 6 \end{array}$$

We can now easily solve (3) for x_3, substitute the result back into (2) and solve for x_2 and so on:

$$\begin{array}{cccccc} (3) & \Longrightarrow & 4x_3 = 2 & \Longrightarrow & x_3 = \frac{1}{2} \\ (2) & \Longrightarrow & 3x_2 + \frac{1}{2} = 5 & \Longrightarrow & x_2 = \frac{3}{2} \\ (1) & \Longrightarrow & 2x_1 + \frac{3}{2} + 3 \times \frac{1}{2} = 1 & \Longrightarrow & x_1 = -1 \end{array}$$

This last step is called "backsolving".

Note that there is an easy way to make sure that we have not made any mechanical errors in deriving this solution — just substitute the purported solution $(-1, 3/2, 1/2)$ back into the original system:

$$\begin{aligned} 2(-1) + \tfrac{3}{2} + 3 \times \tfrac{1}{2} &= 1 \\ 4(-1) + 5\times\tfrac{3}{2} + 7 \times \tfrac{1}{2} &= 7 \\ 2(-1) - 5\times\tfrac{3}{2} + 5 \times \tfrac{1}{2} &= -7 \end{aligned}$$

and verify that each left hand side really is equal to its corresponding right hand side.

Example II.4

$$\begin{aligned} x_1 + 2x_2 + x_3 + 2x_4 + x_5 &= 1 \\ 2x_1 + 4x_2 + 4x_3 + 6x_4 + x_5 &= 2 \\ 3x_1 + 6x_2 + x_3 + 4x_4 + 5x_5 &= 4 \\ x_1 + 2x_2 + 3x_3 + 5x_4 + x_5 &= 4 \end{aligned}$$

has augmented matrix (and check column)

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 1 & 1 \\ 2 & 4 & 4 & 6 & 1 & 2 \\ 3 & 6 & 1 & 4 & 5 & 4 \\ 1 & 2 & 3 & 5 & 1 & 4 \end{array}\right] \begin{array}{c} 8 \\ 19 \\ 23 \\ 16 \end{array}$$

Eliminate the x_1's from equations (2), (3) and (4) as usual

$$\begin{array}{r} (1) \\ (2) - 2(1) \\ (3) - 3(1) \\ (4) - (1) \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 1 & 1 \\ 0 & 0 & 2 & 2 & -1 & 0 \\ 0 & 0 & -2 & -2 & 2 & 1 \\ 0 & 0 & 2 & 3 & 0 & 3 \end{array}\right] \begin{array}{c} 8 \\ 3 \\ -1 \\ 8 \end{array}$$

At this stage x_2 no longer appears in any equation other than the first. So we use equation (2) to eliminate x_3, rather than x_2, from the subsequent equations.

$$\begin{array}{r} (1) \\ (2) \\ (3) + (2) \\ (4) - (2) \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 1 & 1 \\ 0 & 0 & 2 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 3 \end{array}\right] \begin{array}{c} 8 \\ 3 \\ 2 \\ 5 \end{array}$$

Equation (3) no longer contains the variable x_4 at all. So we cannot use equation (3) to eliminate x_4 from other equations. But equation (4) does contain an x_4 and we can use it. To keep the procedure systematic, we exchange equations (3) and (4).

$$\begin{array}{r} (1) \\ (2) \\ (4) \\ (3) \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 1 & 1 \\ 0 & 0 & 2 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} 8 \\ 3 \\ 5 \\ 2 \end{array}$$

If there were more equations we could use the new equation (3) to eliminate x_4 from them. As we are already in a position to backsolve, we don't have to.

We now backsolve to generate the full solution. I will do this in two different ways, that lead to same answer.

I'll call the first method the "direct method". It is the method that we used in Example II.3. We start with the last equation, solving for the last unknown and working backwards.

$$\begin{array}{lcrcr} (4) & \Longrightarrow & x_5 = 1 & & \\ (3) & \Longrightarrow & x_4 + 1 = 3 & \Longrightarrow & x_4 = 2 \\ (2) & \Longrightarrow & 2x_3 + 2 \times 2 - 1 = 0 & \Longrightarrow & x_3 = -\frac{3}{2} \\ (1) & \Longrightarrow & x_1 + 2x_2 - \frac{3}{2} + 2 \times 2 + 1 = 1 & \Longrightarrow & x_1 + 2x_2 = -\frac{5}{2} \end{array}$$

This is now one equation in the two unknowns x_1 and x_2. We can view it as determining x_1 in terms of x_2, with no restriction placed on x_2 at all.

$$\begin{aligned} x_2 &= t, \text{ arbitrary} \\ x_1 &= -\tfrac{5}{2} - 2t \end{aligned}$$

Of course, we could also view x_1 as the free variable with x_2 determined in terms of x_1. Our final answer

$$\begin{aligned}
x_1 &= -\tfrac{5}{2} - 2t \\
x_2 &= t, \text{ arbitrary} \\
x_3 &= -\tfrac{3}{2} \\
x_4 &= 2 \\
x_5 &= 1
\end{aligned}$$

contains one free parameter.

I'll call the second backsolving method the "row reduction method". In it we use row operations to try and convert the equations into the form $x_5 = *$, $x_4 = *$ and so on, where $*$ represents some number. At the present time our augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 1 & 1 \\ 0 & 0 & 2 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} 8 \\ 3 \\ 5 \\ 2 \end{array}$$

Equation (4) is already in the form $x_5 = *$, so there is no need to touch it. We start by using (4) to eliminate all x_5's from equations (1), (2) and (3).

$$\begin{array}{c} (1)-(4) \\ (2)+(4) \\ (3)-(4) \\ (4) \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} 6 \\ 5 \\ 3 \\ 2 \end{array}$$

Equation (3) is now in the form $x_4 = *$, so it is in final form. We now use (3) to eliminate all x_4's from equations (1) and (2).

$$\begin{array}{c} (1)-2(3) \\ (2)-2(3) \\ (3) \\ (4) \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 0 & -4 \\ 0 & 0 & 2 & 0 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} 0 \\ -1 \\ 3 \\ 2 \end{array}$$

Equation (2) is now almost in the form $x_3 = *$, but not quite. The only problem is that the coefficient of x_3 is 2 instead of 1. So we divide equation (2) by 2.

$$\begin{array}{c} (1) \\ (2)/2 \\ (3) \\ (4) \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} 0 \\ -\frac{1}{2} \\ 3 \\ 2 \end{array}$$

and then use (2) to eliminate all x_3's from equation (1).

$$\begin{array}{c} (1)-(2) \\ (2) \\ (3) \\ (4) \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & -5/2 \\ 0 & 0 & 1 & 0 & 0 & -3/2 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} 1/2 \\ -1/2 \\ 3 \\ 2 \end{array}$$

We can now read off the solution from the augmented matrix.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & -5/2 \\ 0 & 0 & 1 & 0 & 0 & -3/2 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} \Longrightarrow \quad x_2 = t,\ x_1 = -\frac{5}{2} - 2t \\ \Longrightarrow \quad x_3 = -\frac{3}{2} \\ \Longrightarrow \quad x_4 = 2 \\ \Longrightarrow \quad x_5 = 1 \end{array}$$

Example II.5 In this example, we just check that $x_1 = -5/2 - 2t,\ x_2 = t,\ x_3 = -3/2,\ x_4 = 2,\ x_5 = 1$ really does solve the system of equations of Example II.4 for all values of t. To do so, we substitute the claimed solution back into the original equations.

$$\begin{aligned}
\big(-\tfrac{5}{2} - 2t\big) + 2t - \tfrac{3}{2} + 2\times 2 + 1 &= 1 \\
2\big(-\tfrac{5}{2} - 2t\big) + 4t + 4\big(-\tfrac{3}{2}\big) + 6\times 2 + 1 &= 2 \\
3\big(-\tfrac{5}{2} - 2t\big) + 6t - \tfrac{3}{2} + 4\times 2 + 5 &= 4 \\
\big(-\tfrac{5}{2} - 2t\big) + 2t + 3\big(-\tfrac{3}{2}\big) + 5\times 2 + 1 &= 4
\end{aligned}$$

Clean up the left hand sides by collecting together all of the constant terms and all of the t terms

$$\begin{aligned}
(-\tfrac{5}{2} - \tfrac{3}{2} + 2\times 2 + 1) + (-2 + 2)t &= 1 \qquad (1') \\
(-2\times\tfrac{5}{2} - 4\times\tfrac{3}{2} + 6\times 2 + 1) + (-2\times 2 + 4)t &= 2 \qquad (2') \\
(-3\times\tfrac{5}{2} - \tfrac{3}{2} + 4\times 2 + 5) + (-3\times 2 + 6)t &= 4 \qquad (3') \\
(-\tfrac{5}{2} - 3\times\tfrac{3}{2} + 5\times 2 + 1) + (-2 + 2)t &= 4 \qquad (4')
\end{aligned}$$

and simplifying

$$\begin{aligned}
1 + 0t &= 1 \\
2 + 0t &= 2 \\
4 + 0t &= 4 \\
4 + 0t &= 4
\end{aligned}$$

The left hand sides do indeed equal the right hand sides for all values of t. In particular, the net coefficient of t on the left hand side of every equation is zero. A better organized way to make this same check, which also tells us something about the nature of the general solution, is the following. Write the solution with all of the unknowns combined into a single vector

$$[x_1, x_2, x_3, x_4, x_5] = \big[-\tfrac{5}{2} - 2t, t, -\tfrac{3}{2}, 2, 1\big]$$

Separate the terms in the solution containing t's from those that don't, using the usual rules for adding vectors and multiplying vectors by numbers.

$$[x_1, x_2, x_3, x_4, x_5] = \big[-\tfrac{5}{2}, 0, -\tfrac{3}{2}, 2, 1\big] + t\big[-2, 1, 0, 0, 0\big]$$

(a) This must be a solution for all t. In particular, when $t = 0$ this must be a solution. So substitution of $[x_1, x_2, x_3, x_4, x_5] = \big[-\tfrac{5}{2}, 0, -\tfrac{3}{2}, 2, 1\big]$ into the left hand sides of the original system (which gives precisely the constant terms on the left hand sides of (1')–(4')) must match the right hand sides of the original system.

(b) Second, I claim that substitution of the **coefficients** of t in the purported general solution (in this example, substitution of $[x_1, x_2, x_3, x_4, x_5] = \big[-2, 1, 0, 0, 0\big]$) into the left hand sides of the original system must yield zero. That is, the coefficients of t in the purported general solution must give a solution of the "associated homogeneous system" (the system you get when you put zeros on the right hand side)

$$\begin{aligned}
x_1 + 2x_2 + x_3 + 2x_4 + x_5 &= 0 \\
2x_1 + 4x_2 + 4x_3 + 6x_4 + x_5 &= 0 \\
3x_1 + 6x_2 + x_3 + 4x_4 + 5x_5 &= 0 \\
x_1 + 2x_2 + 3x_3 + 5x_4 + x_5 &= 0
\end{aligned}$$

To see that this is the case, imagine that $[x_1, x_2, x_3] = [1, 2, 3] + t[4, 5, 6]$ is a solution of the equation $c_1x_1 + c_2x_2 + c_3x_3 = 7$ for all values of t. Then we must have

$$c_1(1 + 4t) + c_2(2 + 5t) + c_3(3 + 6t) = 7$$

or, equivalently,

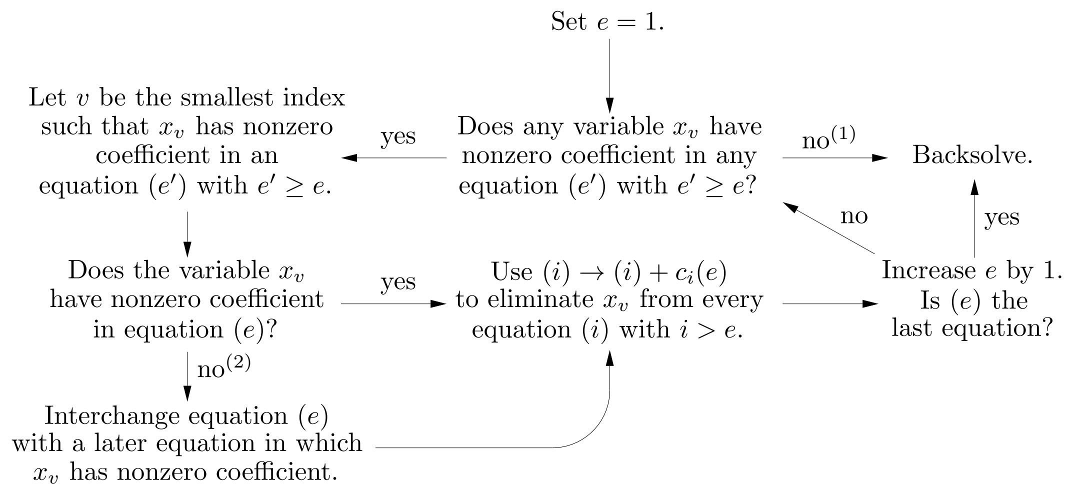
$$(c_1 1 + c_2 2 + c_3 3) + (c_1 4 + c_2 5 + c_3 6)t = 7$$

for all values of t. This forces

$$(c_1 4 + c_2 5 + c_3 6) = 0$$

But $c_1 4 + c_2 5 + c_3 6$ is precisely what you get when you substitute $[x_1, x_2, x_3] = [4, 5, 6]$ into the left hand side of the equation $c_1 x_1 + c_2 x_2 + c_3 x_3 = 7$.

The technique used in Example II.4 is called **Gaussian elimination**. The example was rigged to illustrate every scenario that can arise during execution of the general algorithm. Here is a flow chart showing the general algorithm. We use e to stand for "equation number" and v to stand for "variable number".



Imagine that we are in the midst of applying Gaussian elimination, as in the above flow chart, and that we have finished dealing with rows $1, \cdots, e-1$. These rows will not change during the rest of the elimination process. Denote by M_e the matrix consisting of those rows of the current coefficient matrix having index at least e. For example, if the augmented matrix now looks like

$$\left[\begin{array}{ccc|c} * & * & * & * \\ 0 & 0 & * & * \\ 0 & * & * & * \end{array}\right]$$

and $e = 2$ (in other words, we are about to start work on row 2) then

$$M_e = \begin{bmatrix} 0 & 0 & * \\ 0 & * & * \end{bmatrix}$$

The flow chart (starting with the entry that says "Does any variable x_v have nonzero coefficient in any equation (e') with $e' \geq e$?") now tells us to

- first check to see if M_e is identically zero. For example, if the full augmented matrix is now

$$\left[\begin{array}{ccc|c} * & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \text{or even} \qquad \left[\begin{array}{ccc|c} * & * & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{array}\right]$$

  and $e = 2$, this is the case. If so, we terminate Gaussian elimination and backsolve. This is the branch labelled "no$^{(1)}$" in the flow chart. If not, we
- determine the first variable x_v that has nonzero coefficient in M_e. For example, if $M_e = \begin{bmatrix} 0 & 0 & * \\ 0 & * & * \end{bmatrix}$, then $v = 2$.

- If necessary (this is the branch labelled no$^{(2)}$ in the flow chart), we exchange rows to ensure that x_v has nonzero coefficient in row (e). For example, if $M_e = \begin{bmatrix} 0 & 0 & * \\ 0 & * & * \end{bmatrix}$, we exchange rows. In computer implementations, it is common practice to always move the row with index $e' \ge e$ that has the largest coefficient of x_v into row (e). This is called partial pivotting and is used to reduce the damage caused by round off error.
- We then use row operations to eliminate x_v from all rows below row (e).
- Finally, we increase e by one and, if we are not at the bottom row, repeat.

Exercises for §II.2.

1) Find the general solution of each of the following systems, using the method of Example II.2:

$$\text{(a)}\quad \begin{aligned} x_1 + x_2 &= 1 \\ x_1 - x_2 &= -1 \end{aligned} \qquad \text{b)}\quad \begin{aligned} x_1 + x_2 &= 1 \\ -x_1 - x_2 &= -1 \end{aligned} \qquad \text{c)}\quad \begin{aligned} x_1 + x_2 &= 1 \\ -x_1 - x_2 &= 0 \end{aligned}$$

2) Solve, using Gaussian elimination,

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 2 \\ 2x_1 - 3x_2 + 2x_3 &= 2 \\ 3x_1 + 2x_2 - 4x_3 &= 9 \end{aligned}$$

3) Solve, using Gaussian elimination,

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 6 \\ x_1 - 2x_2 - 2x_3 &= 1 \\ -x_1 + 12x_2 + 8x_3 &= 7 \end{aligned}$$

4) Solve, using Gaussian elimination,

$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 1 \\ x_1 + x_2 + 3x_3 &= 2 \\ 2x_1 + 5x_2 + 9x_3 &= 1 \end{aligned}$$

5) Solve, using Gaussian elimination,

$$\begin{aligned} 3x_1 + x_2 - x_3 + 2x_4 &= 7 \\ 2x_1 - 2x_2 + 5x_3 - 7x_4 &= 1 \\ -4x_1 - 4x_2 + 7x_3 - 11x_4 &= -13 \end{aligned}$$

§II.3 The Form of the General Solution

Example II.3 is typical of most common applications, in that

- the number n of equations is the same as the number m of unknowns,
- the first equation is used to eliminate the first unknown from equations (2) through (n),
- the second equation is used to eliminate the second unknown from equations (3) through (n)
- and so on.

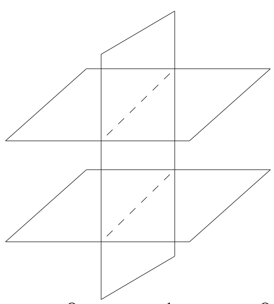
In systems of this type we end up, just before backsolving, with an augmented matrix that looks like

$$\left[\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right]$$

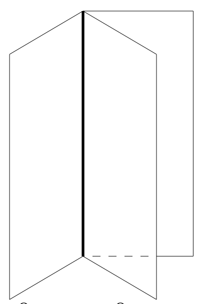
This matrix is triangular. In column number c all entries strictly below row number c are zero. The zeros reflect the fact that variable number c does not appear in equations (c+1) through (n), because equation (c) was used to eliminate the variable x_c from those equations. Backsolving yields a unique value for each unknown.

In Example II.4, we had a system of $n = 4$ equations in $m = 5$ unknowns and ended up with an $m - n = 1$ parameter family of solutions. That is, we could assign an arbitrary value to one of the unknowns (in Example II.4, to x_2). The resulting system of 4 equations in 4 unknowns then had a unique solution. Typically, given n equations in m unknowns, with $m \geq n$, you would expect to be able to assign arbitrary values to $m-n$ of the unknowns and then use the resulting n equations in n unknowns to uniquely determine the remaining n unknowns. That is, typically you expect an $m-n$ parameter family of solutions to a system of n equations in m unknowns.

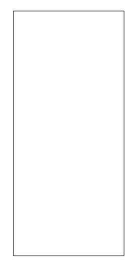
However "typical" does not mean "universal". In fact, the following examples show that all logical possibilities actually occur. Each of the examples has three equations in three unknowns. Each equation determines a plane, which is sketched in the figure accompanying the example. The point (x, y, z) satisfies all three equations if and only if it lies on all three planes. The first example has no solution at all. The second, a 1 parameter family of solutions and the third a 2 parameter family of solutions. The examples have deliberately been chosen so trivial as to look silly. In the real world they tend to arise in highly disguised form, in which they do not look at all silly.



$x_3 = 0,\ x_3 = 1,\ x_1 = 0$
No solutions



$x_1 = 0,\ x_2 = 0,\ x_1 + x_2 = 0$
Solution: $x_1 = x_2 = 0$
$x_3 = \alpha$, arbitrary



$x_2 = 0,\ 2x_2 = 0,\ 3x_2 = 0$
Solution: $x_1 = \alpha$, arbitrary
$x_2 = 0$
$x_3 = \beta$, arbitrary

There is a fourth example which has a 3 parameter family of solutions, namely $0x_1 = 0,\ 0x_2 = 0,\ 0x_3 = 0$, which has general solution $x_1 = \alpha,\ x_2 = \beta,\ x_3 = \gamma$ with all of $\alpha,\ \beta,\ \gamma$ arbitrary.

Imagine that you are solving a linear system of equations. You have applied Gaussian elimination and are about to start backsolving. Denote by m_r the index of the first nonzero entry in row r. (We may as well throw out rows that have no nonzero entries.) In Example II.3, $m_r = r$ for all r. But, this is not always the case. In Example II.4, the final augmented matrix before backsolving was

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 1 & 1 \\ 0 & 0 & 2 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

In this example $m_1 = 1,\ m_2 = 3,\ m_3 = 4,\ m_4 = 5$. After eliminating x_1 from equations (2) through (4), we discovered that there were no x_2's in any of the equations (2) through (4). So we ignored variable x_2 and used equation (2) to eliminate x_3, rather than x_2, from equations (3) and (4). That is why, in this example, $m_2 > 2$.

The Gaussian elimination algorithm always yields an augmented matrix with

$$m_1 < m_2 < m_3 \cdots$$

The equation (j) that we use when backsolving is of the form

$$ax_{m_j} + \sum_{n > m_j} b_n x_n = c$$

for some constants $a,\ b_n,\ c$. Furthermore the coefficient a is always nonzero, because m_j is the column number of the first nonzero coefficient in row j. Any of the variables $x_n,\ n > m_j$, that have not already been assigned values in earlier backsolving steps may be assigned arbitrary values. The equation $ax_{m_j} + \sum_{n>m_j} b_n x_n = c$ then uniquely determines x_{m_j}. Consequently, all variables other than $x_{m_1}, x_{m_2}, \cdots$ are left as arbitrary parameters. In Example II.4,

$$
\begin{array}{lllll}
(4) & \Longrightarrow & x_{m_4} = 1 & & \\
(3) & \Longrightarrow & x_{m_3} + 1 = 3 & \Longrightarrow & x_{m_3} = 2 \\
(2) & \Longrightarrow & 2x_{m_2} + 2 \times 2 - 1 = 0 & \Longrightarrow & x_{m_2} = -\frac{3}{2} \\
(1) & \Longrightarrow & x_{m_1} + 2x_2 - \frac{3}{2} + 2 \times 2 + 1 = 1 & \Longrightarrow & x_{m_1} + 2x_2 = -\frac{5}{2} \\
 & & & \Longrightarrow & x_2 = c,\ \text{arbitrary} \\
 & & & & x_{m_1} = -\frac{5}{2} - 2c
\end{array}
$$

So the number of free parameters in the solution is the number of variables that are not x_{m_i}'s, which in this case is one. The general solution is

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -5/2 - 2c \\ c \\ -3/2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 0 \\ -3/2 \\ 2 \\ 1 \end{bmatrix} + c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

We are viewing $x_1,\ x_2, \cdots,\ x_5$ as the components of a five dimensional vector. In anticipation of some definitions that will be introduced in the next chapter, we choose to write the vector as a column, rather than a row, inside square brackets.

As a second example, supposed that application of Gauss reduction to some system of equations yields

$$
\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]
$$

In this example $m_1 = 1,\ m_2 = 2$ and backsolving gives

$$
\begin{array}{lllll}
(2) & \Longrightarrow & x_{m_2} + x_3 + x_4 = 1 & \Longrightarrow & x_{m_2} = 1 - x_3 - x_4 \\
 & & & & x_4 = c_1,\ \text{arbitrary} \\
 & & & & x_3 = c_2,\ \text{arbitrary} \\
 & & & & x_2 = 1 - c_1 - c_2 \\
(1) & \Longrightarrow & x_{m_1} + (1 - c_1 - c_2) + c_2 + c_1 = 1 & \Longrightarrow & x_1 = x_{m_1} = 0
\end{array}
$$

The general solution is

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 - c_1 - c_2 \\ c_2 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}
$$

Definition II.6 The **rank** of a matrix is the number of nonzero rows in the triangular matrix that results from Gaussian elimination. That is, if the triangular matrix $[R]$ can be arrived at by applying a sequence of row operations to the matrix $[A]$, then the rank of $[A]$ is the number of nonzero rows in $[R]$.

Consider any system of linear equations. Denote by $[\,A \,|\, b\,]$ the augmented matrix of the system. In particular $[\,A\,]$ is the coefficient matrix of the system. There are three possibilities.

Possibility 1: rank $[\,A\,] <$ rank $[\,A\,|\,b\,]$

This possibility is illustrated by

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The rank of the coefficient matrix, that is the number of nonzero rows to the left of the vertical bar, is two while the rank of the augmented matrix is three. The third row of the augmented matrix stands for $0x_1 + 0x_2 + 0x_3 = 1$, which can never be true. When rank $[\,A\,] <$ rank $[\,A\,|\,b\,]$, the linear system has no solution at all.

Possibility 2: #unknowns = rank $[\,A\,] =$ rank $[\,A\,|\,b\,]$

This possibility is illustrated by

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

for which the number of unknowns, the rank of the coefficient matrix and the rank of the augmented matrix are all equal to three. Every variable is an x_{m_i}, so the general solution contains no free parameters. There is exactly one solution.

Possibility 3: #unknowns $>$ rank $[\,A\,] =$ rank $[\,A\,|\,b\,]$

This possibility is illustrated by

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

for which the number of unknowns is four while the rank of the coefficient matrix and the rank of the augmented matrix are both 2. The number of variables that are x_{m_i}'s is $\rho = \text{rank}\,[A]$, so the number of free parameters in the general solution is $n - \rho$, the number of unknowns minus rank $[A]$. In particular, the system has infinitely many solutions. If we use $\vec{x}$ to denote a column vector with components $x_1, \cdots, x_n$, the general solution is of the form

$$\vec{x} = \vec{u} + c_1\vec{v}_1 + \cdots + c_{n-\rho}\vec{v}_{n-\rho}$$

where

- $c_1, \cdots, c_{n-\rho}$ are the arbitrary constants.
- The component u_i of $\vec{u}$ is the term in the x_i component of the general solution that is not multiplied by any arbitrary constant. $\vec{u}$ is called a particular solution of the system. It is the solution you get when you set all of the arbitrary constants to zero.
- Each component of $\vec{v}_j$ is the coefficient of c_j in the corresponding component of the general solution. We claim that substitution of $\vec{v}_j$ into the left hand side of each equation in the original system must yield zero. This is because when $\vec{x} = \vec{u} + c_j\vec{v}_j$ is substituted back into the left hand side of the original system of equations (as was done in Example II.5), the resulting left hand sides must match the corresponding right hand sides for all values of c_j. This forces the net coefficient of c_j in each resulting left hand side to be zero.

Example II.7 The coefficient and augmented matrices for the system

$$\begin{aligned} x_1 + \ \ x_2 + \ \ x_3 + \ \ x_4 &= 2 \\ 2x_1 + 3x_2 + 4x_3 + 2x_4 &= 5 \\ 3x_1 + 4x_2 + 5x_3 + 3x_4 &= 7 \end{aligned}$$

are

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 2 \\ 3 & 4 & 5 & 3 \end{bmatrix} \qquad \text{and} \qquad \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 2 & 3 & 4 & 2 & 5 \\ 3 & 4 & 5 & 3 & 7 \end{array}\right]$$

respectively. The standard Gaussian elimination row operations

$$\begin{array}{c} (1) \\ (2)-2(1) \\ (3)-3(1) \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 \end{array}\right] \qquad \begin{array}{c} (1) \\ (2) \\ (3)-(2) \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

replace the coefficient and augmented matrices with

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

respectively, both of which have two rows that contain nonzero entries. So the coefficient and augmented matrices both have rank 2. As there are four unknowns, $x_1, \cdots, x_4$, the general solution should contain $4-2=2$ free parameters. Backsolving

$$\begin{array}{lllll} (2) & \implies & x_2 + 2x_3 = 1 & \implies & x_2 = 1 - 2x_3, \text{ with } x_3 = c_1 \text{ arbitrary} \\ (1) & \implies & x_1 + (1-2c_1) + c_1 + x_4 = 2 & \implies & x_1 = 1 + c_1 - x_4, \text{ with } x_4 = c_2 \text{ arbitrary} \end{array}$$

confirms that there are indeed two free parameters in the general solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1+c_1-c_2 \\ 1-2c_1 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

If we have made no mechanical errors, we should have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \implies \qquad \begin{aligned} x_1 + x_2 + x_3 + x_4 &= 2 \\ 2x_1 + 3x_2 + 4x_3 + 2x_4 &= 5 \\ 3x_1 + 4x_2 + 5x_3 + 3x_4 &= 7 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \implies \qquad \begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ 2x_1 + 3x_2 + 4x_3 + 2x_4 &= 0 \\ 3x_1 + 4x_2 + 5x_3 + 3x_4 &= 0 \end{aligned}$$

We do.

Exercises for §II.3

1) Consider the three planes

$$x + 2y + 5z = 7 \qquad 2x - y = -1 \qquad 2x + y + 4z = k$$

   a) For which values of the parameter k do these three planes have at least one point in common?
   a) Determine the common points.

2) A student spends a total of 31 hours per week studying Algebra, Biology, Calculus and Economics. The student devotes 5 more hours to Algebra than to Biology and Economics combined and 3 fewer hours to Calculus than to Algebra and Biology combined. What is the maximum number of hours that can be devoted Economics?

3) Let $a,\ b,\ c,\ d,\ \alpha$ and β be fixed real numbers. Consider the system of equations

$$\begin{aligned} ax + by &= \alpha \\ cx + dy &= \beta \end{aligned}$$

   For which values of $a,\ b,\ c,\ d,\ \alpha$ and β is there at least one solution? For which values of $a,\ b,\ c,\ d,\ \alpha$ and β is there exactly one solution?

§II.4 Homogeneous Systems

A system of linear equations is homogeneous if all of the constant terms are zero. That is, in the notation of Definition II.1, $b_1 = b_2 = \cdots = b_m = 0$. For example, if we replace all of the right hand sides in Example II.4 by zeros, we get the homogeneous system

$$\begin{aligned} x_1 + 2x_2 + \ \ x_3 + 2x_4 + \ \ x_5 &= 0 \\ 2x_1 + 4x_2 + 4x_3 + 6x_4 + \ \ x_5 &= 0 \\ 3x_1 + 6x_2 + \ \ x_3 + 4x_4 + 5x_5 &= 0 \\ x_1 + 2x_2 + 3x_3 + 5x_4 + \ \ x_5 &= 0 \end{aligned}$$

For a homogeneous system, the last column of the augmented matrix is filled with zeros. Any row operation applied to the augmented matrix has no effect at all on the last column. So, the final column of the row reduced augmented matrix $[\,R\,|\,d\,]$ that results from Gaussian elimination is still filled with zeros. Consequently, the rank of $[\,A\,|\,b\,]$ (i.e. the number of nonzero rows in $[\,R\,|\,d\,]$) is the same as the rank of $[A]$ (i.e. the number of nonzero rows in $[R]$) and Possibility 1 of §II.3 cannot occur. Homogeneous systems always have at least one solution. This should be no surprise: $x_1 = x_2 = \cdots = x_n = 0$ is always a solution. For homogeneous systems there are only two possibilities

$$\begin{aligned} \text{rank}\,[A] = \#\text{unknowns} \quad &\Longrightarrow \quad \text{the only solution is } x_1 = x_2 = \cdots = x_n = 0 \\ \text{rank}\,[A] < \#\text{unknowns} \quad &\Longrightarrow \quad \#\text{free parameters} = \#\text{unknowns} - \text{rank}\,[A] > 0 \\ &\Longrightarrow \quad \text{there are infinitely many solutions.} \end{aligned}$$

Exercises for §II.4.

1) Let $a,\ b,\ c$ and d be fixed real numbers. Consider the system of equations

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \end{aligned}$$

There is always at least one solution to this system, namely $x = y = 0$. For which values of $a,\ b,\ c$ and d is there at exactly one other solution? For which values of $a,\ b,\ c$ and d is there at least one other solution?

2) Show that if $\vec{x}$ and $\vec{w}$ are any two solutions of a linear system of equations, then $\vec{x} - \vec{w}$ is a solution of the associated homogeneous system. Show that, if $\vec{u}$ is any solution of the original system, every solution is of the form $\vec{u} + \vec{v}$, where $\vec{v}$ is a solution of the associated homogeneous system.

3 a) Show that if $\vec{v}_1$ and $\vec{v}_2$ are both solutions of a given homogeneous system of equations and if c_1 and c_2 are numbers then $c_1\vec{v}_1 + c_2\vec{v}_2$ is also a solution.

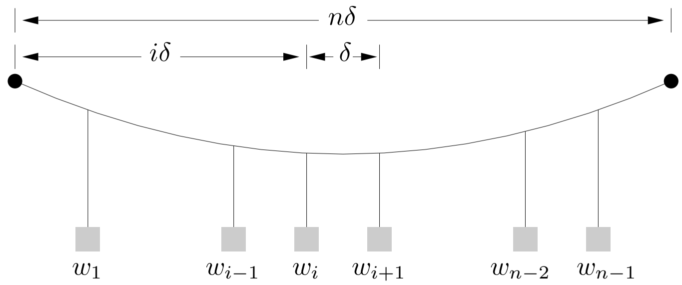
b) Show that if $\vec{v}_1,\ \cdots,\ \vec{v}_k$ are all solutions of a given homogeneous system of equations and if $c_1,\ \cdots,\ c_k$ are numbers then $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k$ is also a solution.

4) Express the general solution of the system of equations given in Exercise 5 of §II.2 in the form $\vec{x} = \vec{u} + c_1\vec{v}_1 + c_2\vec{v}_2$ with the first component of $\vec{v}_1$ being 0 and the second component of $\vec{v}_2$ being 0. Express the general solution in the form $\vec{x} = \vec{u} + d_1\vec{w}_1 + d_2\vec{w}_2$ with the first two components of $\vec{w}_1$ being equal and the first two components of $\vec{w}_2$ being negatives of each other. Express $\vec{w}_1$ and $\vec{w}_2$ in terms of $\vec{v}_1$ and $\vec{v}_2$. Express c_1 and c_2 in terms of d_1 and d_2.

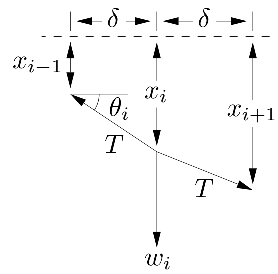
§II.5 The Loaded Cable

The loaded cable is a typical of a large class of physical systems which can, under suitable conditions, be well described by a linear system of equations. Consider a cable stretched between two points a distance $n\delta$

apart. Suppose that $n-1$ weights, $w_1,\ w_2,\cdots,\ w_{n-1}$, are hung at regular intervals along the cable. These weights could form a bridge for example, or they could be used to approximate a continuous loading of the cable.



Assume that the cable is in equilibrium, the weight of the cable itself may be neglected (or is included in the w_i's) and that the cable has not been distorted too much away from horizontal. Then the tension T in the cable will be (essentially) uniform along its length and there must be no net vertical component of force acting on the point from which w_i is hung. There are three forces acting on that point. The first is



the tension in the part of the cable between w_{i-1} and w_i. This tension pulls to the left in the direction of that part of the cable. If that part of the cable is at an angle θ_i below horizontal the vertical component of its tension has magnitude $T\sin\theta_i$ and points upward. The second force is the tension in the part of the cable between w_i and w_{i+1}. This tension pulls to the right in the direction of that part of the cable. If that part of the cable is at an angle θ_{i+1} below horizontal the vertical component of its tension has magnitude $T\sin\theta_{i+1}$ and points downward. The third force is the weight w_i itself, which acts downward. So to have no net vertical force we need

$$T\sin\theta_i = w_i + T\sin\theta_{i+1}$$

Fix some horizontal line and define x_i to be the vertical distance from the horizontal line to the point on the cable at which w_i is attached. Then $\tan\theta_i = (x_i - x_{i-1})/\delta$. Thanks to the "small amplitude" hypothesis, $\cos\theta_i \approx 1$ so that $\sin\theta_i \approx \frac{\sin\theta_i}{\cos\theta_i} = \tan\theta_i = \frac{x_i - x_{i-1}}{\delta}$ so that the force balance equation approximates to

$$T\frac{x_i - x_{i-1}}{\delta} = w_i + T\frac{x_{i+1} - x_i}{\delta}$$

which in turn simplifies to

$$T\frac{x_{i-1} - 2x_i + x_{i+1}}{\delta} = -w_i$$
$$\implies \quad x_{i-1} - 2x_i + x_{i+1} = -\frac{w_i\delta}{T}$$

Suppose that the left and right hand ends of the cable are fastened at height 0. Then the last equation even applies at the ends $i = 1,\ n-1$ if we set $x_0 = x_n = 0$. Thus the system of equations determining the cable configuration is

$$x_0 = 0,\ x_{i-1} - 2x_i + x_{i+1} = -\frac{w_i\delta}{T} \text{ for } i = 1,\cdots,n-1\ ,\ x_n = 0$$

In particular, when $n = 5$

$$\begin{aligned}
x_0 \qquad\qquad\qquad\qquad &= 0 \\
x_0 - 2x_1 + \quad x_2 \qquad\qquad &= -\frac{w_1\delta}{T} \qquad (i=1) \\
x_1 - 2x_2 + \quad x_3 \qquad &= -\frac{w_2\delta}{T} \qquad (i=2) \\
x_2 - 2x_3 + \quad x_4 &= -\frac{w_3\delta}{T} \qquad (i=3) \\
x_3 - 2x_4 + x_5 &= -\frac{w_4\delta}{T} \qquad (i=4) \\
x_5 &= 0
\end{aligned}$$

or, if we simplify by substituting $x_0 = x_5 = 0$ into the remaining equations

$$\begin{aligned}
-2x_1 + \quad x_2 \qquad\qquad &= -\frac{w_1\delta}{T} \qquad (i=1) \\
x_1 - 2x_2 + \quad x_3 \qquad &= -\frac{w_2\delta}{T} \qquad (i=2) \\
x_2 - 2x_3 + \quad x_4 &= -\frac{w_3\delta}{T} \qquad (i=3) \\
x_3 - 2x_4 &= -\frac{w_4\delta}{T} \qquad (i=4)
\end{aligned}$$

As a concrete example, suppose that $w_j\delta/T = 1$ for all j. Then the augmented matrix is

$$\left[\begin{array}{cccc|c} -2 & 1 & 0 & 0 & -1 \\ 1 & -2 & 1 & 0 & -1 \\ 0 & 1 & -2 & 1 & -1 \\ 0 & 0 & 1 & -2 & -1 \end{array}\right] \begin{array}{c} -2 \\ -1 \\ -1 \\ -2 \end{array}$$

Gaussian elimination gives

$$\begin{array}{c} (1) \\ (2)+\frac{1}{2}(1) \\ (3) \\ (4) \end{array} \left[\begin{array}{cccc|c} -2 & 1 & 0 & 0 & -1 \\ 0 & -\frac{3}{2} & 1 & 0 & -\frac{3}{2} \\ 0 & 1 & -2 & 1 & -1 \\ 0 & 0 & 1 & -2 & -1 \end{array}\right] \begin{array}{c} -2 \\ -2 \\ -1 \\ -2 \end{array}$$

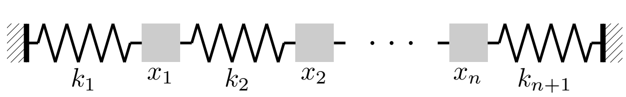
$$\begin{array}{c} (1) \\ (2) \\ (3)+\frac{2}{3}(2) \\ (4) \end{array} \left[\begin{array}{cccc|c} -2 & 1 & 0 & 0 & -1 \\ 0 & -\frac{3}{2} & 1 & 0 & -\frac{3}{2} \\ 0 & 0 & -\frac{4}{3} & 1 & -2 \\ 0 & 0 & 1 & -2 & -1 \end{array}\right] \begin{array}{c} -2 \\ -2 \\ -7/3 \\ -2 \end{array}$$

$$\begin{array}{c} (1) \\ (2) \\ (3) \\ (4)+\frac{3}{4}(3) \end{array} \left[\begin{array}{cccc|c} -2 & 1 & 0 & 0 & -1 \\ 0 & -\frac{3}{2} & 1 & 0 & -\frac{3}{2} \\ 0 & 0 & -\frac{4}{3} & 1 & -2 \\ 0 & 0 & 0 & -\frac{5}{4} & -\frac{5}{2} \end{array}\right] \begin{array}{c} -2 \\ -2 \\ -7/3 \\ -11/4 \end{array}$$

and backsolving gives $x_4 = 2$, $x_3 = 3$, $x_2 = 3$ and $x_1 = 2$.

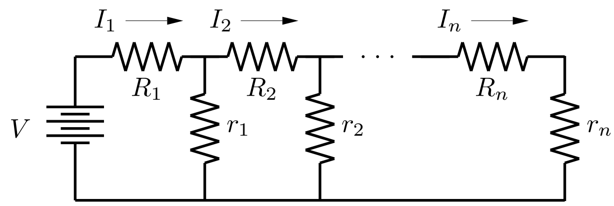
Exercises for §II.5.

1) Consider a system of n masses coupled by springs as in the figure



The masses are constrained to move horizontally. The distance from mass number j to the left hand wall is x_j. The j^{th} spring has natural length ℓ_j and spring constant k_j. According to Hooke's law (which Hooke published as an anagram in 1676 – he gave the solution to the anagram in 1678) the force exerted by spring number j is k_j times the extension of spring number j, where the extension of a spring is its length minus its natural length. The distance between the two walls is L. Give the system of equations that determine the equilibrium values of $x_1, \cdots, x_j$. (This problem will get more interesting once we introduce time dependence later in the course. Then, for large n, the system models a bungee chord.)

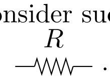
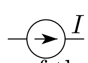
2) Consider the electrical network in the figure



Assume that the DC voltage V is given and that the resistances $R_1, \cdots, R_n$ and $r_1, \cdots, r_n$ are given. Find the system of equations that determine the currents $I_1, \cdots, I_n$. You will need the following experimental facts. (a) The voltage across a resistor of resistance R that is carrying current I is IR. (b) The net current entering any node of the circuit is zero. (c) The voltage between any two points is independent of the path used to travel between the two points. (This problem will get more interesting once we have introduced time dependence, capacitors and inductors, later in the course.)

§II.6 Resistor Networks

We now give a more systematic treatment of circuit problems like that in Exercise 2 of §II.5. By definition, a resistor network is an electrical circuit that consists only of wires, resistors, voltage sources and current sources. Resistor networks are a special case of the much more interesting and much more useful "linear circuits". Linear circuits may contain capacitors and inductors as well as resistors and voltage and current sources. We'll consider such circuits in §IV.

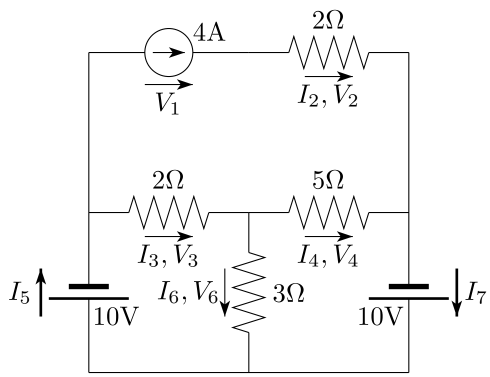
- A resistor is drawn  . If the current through the resistor is i amps and the resistance of the resistor is R Ohms, then the voltage *drop* across the resistor (in the direction of the current) is $v = iR$.
- A voltage source is drawn  . The voltage *increase* across the source, from the short side to the long side, is always V, regardless of the current flowing through the source.
- A current source is drawn  . The current flowing through the source, in the direction of the arrow, is always I, regardless of the voltage across the source.

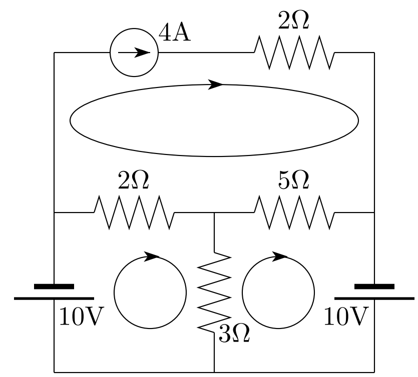
In addition to these circuit element characterstics, there two "conservation laws" which determine the behaviour of resistor networks.

- *Kirchhoff's current law* says that the sum of all currents flowing into any junction in the circuit must equal the sum of all currents flowing out of that same junction.
- *Kirchhoff's voltage law* says that the sum of all voltage drops around any closed loop in the circuit must be zero.

Here is an example which gives all of the equations arising from the application of these circuit element laws and Kirchhoff's laws to a specific circuit. We won't solve any equations in this example, because there is a more efficient way to formulate the equations that we'll get to shortly.

Example II.8 Consider the resistor network of





Note that here

- ∘ V_2, V_3, V_4 and V_6 are the voltage **drops** in the direction of the arrow while
- ∘ V_1 is the voltage **increase** in the direction of the arrow.

The circuit element laws for this circuit are

$$I_2 = 4 \qquad V_2 = 2I_2 \qquad V_3 = 2I_3 \qquad V_4 = 5I_4 \qquad V_6 = 3I_6$$

Kirchoff's current laws for the circuit are

$$I_5 = I_3 + 4 \qquad I_3 = I_4 + I_6 \qquad I_2 + I_4 = I_7 \qquad I_6 + I_7 = I_5$$

Kirchhoff's voltage laws for the three loops in the figure on the right above are

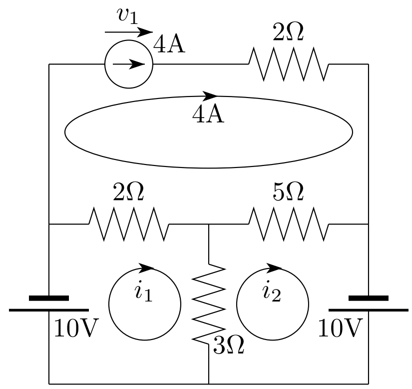
$$-V_1 + V_2 - V_4 - V_3 = 0 \qquad 10 + V_3 + V_6 = 0 \qquad -V_6 + V_4 - 10 = 0$$

To get the signs right in such equations, pretend that you are walking around a loop in the direction specified for the loop and record the voltage *drop* the you encounter as you pass through each circuit element. For example, for the first loop,

- ∘ V_1 was defined to be the voltage *increase* in the direction of the loop and so contributes $-V_1$ to the equation
- ∘ V_2 was defined to be the voltage *drop* in the direction of the loop and so contributes $+V_2$ to the equation
- ∘ V_4 was defined to be the voltage *drop* the direction *opposite* to the loop and so contributes $-V_4$ to the equation and
- ∘ V_3 was defined to be the voltage *drop* the direction *opposite* to the loop and so contributes $-V_3$ to the equation.

We have found twelve equations in the unkowns I_2, I_3, I_4, I_5, I_6, I_7, V_1, V_2, V_4 and V_6. Of course many of the equations are pretty trivial. There is a more efficient way to formulate the equations that substantially reduces the number of equations and unknowns and, in particular, eliminates the trivial equations. We consider it in the next example.

Example II.9 In this example, we apply the method of current loops to formulate the equations for the circuit of Example II.8. To do so, we select a complete set of loops for the circuit as in the right hand figure of Example II.8. We imagine that there is a current flowing around each of the loops. Because of the current source, the current flowing in the top loop must be 4A. We just give names, say i_1 and i_2 to the currents in the other two loops. In the notation of Example II.8, the net current flowing through the 3Ω resistor,



for example, is $I_6 = i_1 - i_2$. Because the currents i_1, i_2 and 4A are flowing around closed loops, Kirchhoff's current laws are automatically satisfied. Kirchhoff's voltage laws are

$$-V_1 + 2\times 4 + 5(4-i_2) + 2(4-i_1) = 0 \quad 10 + 2(i_1-4) + 3(i_1-i_2) = 0 \quad 3(i_2-i_1) + 5(i_2-4) - 10 = 0$$

or

$$\begin{aligned} V_1 + 2i_1 + 5i_2 &= 36 \\ 5i_1 - 3i_2 &= -2 \\ -3i_1 + 8i_2 &= 30 \end{aligned}$$

This gives the augmented matrix and upper triangular form

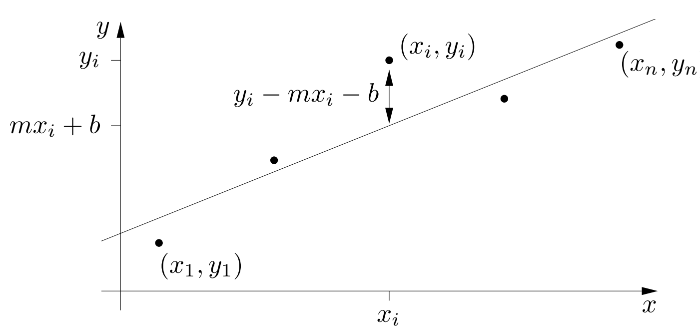
$$\left[\begin{array}{ccc|c} 1 & 2 & 5 & 36 \\ 0 & 5 & -3 & -2 \\ 0 & -3 & 8 & 30 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 2 & 5 & 36 \\ 0 & 5 & -3 & -2 \\ 0 & 0 & 31 & 144 \end{array}\right] \begin{matrix} \\ \\ 5(3)+3(2) \end{matrix}$$

Backsolving gives

$$i_2 = \tfrac{144}{31} = 4.6452A \qquad i_1 = \tfrac{1}{5}\big(-2 + 3\times\tfrac{144}{31}\big) = \tfrac{370}{155} = 2.3871A \qquad V_1 = 36 - 2\times\tfrac{370}{155} - 5\times\tfrac{144}{31} = 8V$$

§II.7 Linear Regression

Imagine an experiment in which you measure one quantity, call it y, as a function of a second quantity, say x. For example, y could be the current that flows through a resistor when a voltage x is applied to it. Suppose that you measure n data points $(x_1, y_1), \cdots, (x_n, y_n)$ and that you wish to find the straight line $y = mx + b$ that fits the data best. If the data point (x_i, y_i) were to land exactly on the line $y = mx + b$



we would have $y_i = mx_i + b$. If it doesn't land exactly on the line, the vertical distance between (x_i, y_i) and the line $y = mx + b$ is $|y_i - mx_i - b|$. That is the discrepancy between the measured value of y_i and the corresponding idealized value on the line is $|y_i - mx_i - b|$. One measure of the total discrepancy for all data points is $\sum_{i=1}^n |y_i - mx_i - b|$. A more convenient measure, which avoids the absolute value signs, is

$$D(m, b) = \sum_{i=1}^n (y_i - mx_i - b)^2$$

We will now find the values of m and b that give the minimum value of $D(m, b)$. The corresponding line $y = mx + b$ is generally viewed as the line that fits the data best.

You learned in your first Calculus course that the value of m that gives the minimum value of a function of one variable $f(m)$ obeys $f'(m) = 0$. The analogous statement for functions of two variables is the following. First pretend that b is just a constant and compute the derivative of $D(m, b)$ with respect to m. This is called the partial derivative of $D(m, b)$ with respect to m and denoted $\frac{\partial D}{\partial m}(m, b)$. Next pretend that m is just a constant and compute the derivative of $D(m, b)$ with respect to b. This is called the partial derivative of $D(m, b)$ with respect to b and denoted $\frac{\partial D}{\partial b}(m, b)$. If (m, b) gives the minimum value of $D(m, b)$, then

$$\tfrac{\partial D}{\partial m}(m, b) = \tfrac{\partial D}{\partial b}(m, b) = 0$$

For our specific $D(m, b)$

$$\begin{aligned} \tfrac{\partial D}{\partial m}(m, b) &= \sum_{i=1}^n 2(y_i - mx_i - b)(-x_i) \\ \tfrac{\partial D}{\partial b}(m, b) &= \sum_{i=1}^n 2(y_i - mx_i - b)(-1) \end{aligned}$$

It is important to remember here that all of the x_i's and y_i's here are given numbers. The only unknowns are m and b. The two partials are of the forms

$$\begin{aligned} \tfrac{\partial D}{\partial m}(m, b) &= 2c_{xx}m + 2c_x b - 2c_{xy} \\ \tfrac{\partial D}{\partial b}(m, b) &= 2c_x m + 2nb - 2c_y \end{aligned}$$

where the various c's are just given numbers whose values are

$$c_{xx} = \sum_{i=1}^n x_i^2 \qquad c_x = \sum_{i=1}^n x_i \qquad c_{xy} = \sum_{i=1}^n x_i y_i \qquad c_y = \sum_{i=1}^n y_i$$

So the value of (m, b) that gives the minimum value of $D(m, b)$ is determined by

$$\begin{aligned} c_{xx}m + c_x b &= c_{xy} \qquad (1) \\ c_x m + nb &= c_y \qquad (2) \end{aligned}$$

This is a system of two linear equations in the two unknowns m and b, which is easy to solve:

$$\begin{aligned} &n(1) - c_x(2): \quad [nc_{xx} - c_x^2]m = nc_{xy} - c_x c_y \quad \Longrightarrow \quad m = \frac{nc_{xy} - c_x c_y}{nc_{xx} - c_x^2} \\ &c_x(1) - c_{xx}(2): \quad [c_x^2 - nc_{xx}]b = c_x c_{xy} - c_{xx} c_y \quad \Longrightarrow \quad b = \frac{c_{xx} c_y - c_x c_{xy}}{nc_{xx} - c_x^2} \end{aligned}$$

Exercises for §II.6.

1) Consider the problem of finding the parabola $y = ax^2 + mx + b$ which best fits the n data points (x_1, y_1), $\cdots$, (x_n, y_n). Derive the system of three linear equations which determine a, m, b. Do not attempt to solve them.

§II.8 Worked Problems

Questions

1) For each of the following systems of linear equations, find the general solution in parametric form. Give a geometric interpretation.

(a) $\begin{aligned} x + 4y &= 5 \\ 2x + 7y &= 6 \end{aligned}$

(b) $\begin{aligned} 2x + 4y &= -12 \\ 6x + 12y &= -24 \end{aligned}$

(c) $\begin{aligned} x + 3y &= 3 \\ 2x + 6y &= 6 \end{aligned}$

(d) $\begin{aligned} 3x + 2y &= 1 \\ 4x + 5y &= 6 \end{aligned}$

2) For each of the following systems of linear equations, find the general solution in parametric form. Give a geometric interpretation.

(a) $\begin{aligned} x + 3y + 4z &= 2 \\ 3x + 8y + 12z &= 5 \end{aligned}$

(b) $\begin{aligned} 3x + 2y - z &= -2 \\ 6x + 4y - 2z &= 5 \end{aligned}$

(c) $\begin{aligned} x + 2y + 3z &= 2 \\ 4x + 10y + 6z &= 4 \end{aligned}$

(d) $\begin{aligned} 2x + 3y + z &= 0 \\ 4x + 6y + 3z &= 8 \end{aligned}$

3) For each of the following systems of linear equations, find the general solution in parametric form. Give a geometric interpretation.

(a) $\begin{aligned} x \qquad + 4z &= 1 \\ 2x + 2y + 4z &= 2 \\ 2x - 2y + 8z &= 6 \end{aligned}$

(b) $\begin{aligned} x + 2y + 4z &= 1 \\ 2x + 3y + 7z &= 5 \\ 5x + 5y + 15z &= 9 \end{aligned}$

(c) $\begin{aligned} 3x + 4y + 2z &= 1 \\ 6x + 12y + 5z &= 4 \\ 3x + 8y + 3z &= 3 \end{aligned}$

(d) $\begin{aligned} x + y + z &= 2 \\ x + 2y - z &= 4 \\ x + y + 2z &= 4 \end{aligned}$

(e) $\begin{aligned} x + 3y + 3z &= 5 \\ 2x + 6y + 3z &= 7 \\ x + 3y - 3z &= 2 \end{aligned}$

4) Find the general solution for each of the following systems of linear equations.

(a) $\begin{aligned} x_1 + 3x_2 + x_3 + x_4 &= 1 \\ -2x_1 - 5x_2 - x_3 - 5x_4 &= 1 \end{aligned}$

(b) $\begin{aligned} x_1 + 3x_2 + 2x_3 + x_4 &= 1 \\ x_1 + 3x_2 + x_3 + 2x_4 &= 1 \\ x_1 + 4x_2 + x_3 + 3x_4 &= 2 \end{aligned}$

(c) $\begin{aligned} x_1 + 3x_2 + x_3 + 7x_4 &= 5 \\ 3x_1 + 9x_2 + x_3 + 11x_4 &= 17 \\ 2x_1 + 6x_2 + 2x_3 + 14x_4 &= 10 \end{aligned}$

(d) $\begin{aligned} x_1 + 5x_2 + 3x_3 + 2x_4 + 3x_5 &= 1 \\ 2x_1 + 10x_2 + 6x_3 + 5x_4 + 5x_5 &= 4 \end{aligned}$

(e) $\begin{aligned} 2x_1 + 4x_2 + 2x_3 + 2x_4 + x_5 &= 18 \\ 4x_1 + 8x_2 + 6x_3 \qquad + 3x_5 &= 44 \\ -2x_1 - 4x_2 + 2x_3 - 10x_4 + 2x_5 &= 2 \end{aligned}$

(f) $\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 14 \\ x_1 + x_2 - x_3 - x_4 &= -4 \\ x_1 - x_2 + x_3 - x_4 &= -2 \\ x_1 - x_2 - x_3 + x_4 &= 0 \end{aligned}$

(g) $\begin{aligned} x_1 \qquad + x_3 + 3x_4 &= 2 \\ x_1 + 2x_2 + 3x_3 + 2x_4 &= 4 \\ 3x_1 + 2x_2 + 7x_3 \qquad &= 1 \\ x_1 \qquad + 3x_3 - 5x_4 &= 8 \end{aligned}$

(h) $\begin{aligned} x_1 + 2x_2 + x_3 + 2x_4 + 3x_5 &= 0 \\ 2x_1 + 5x_2 + 2x_3 + 5x_4 + 7x_5 &= 1 \\ x_1 + 3x_2 + x_3 + 3x_4 + 4x_5 &= 1 \\ x_1 + x_2 + x_3 + x_4 + x_5 &= 0 \end{aligned}$

5) Construct the augmented matrix for each of the following linear systems of equations. In each case determine the rank of the coefficient matrix, the rank of the augmented matrix, whether or not the system has any solutions, whether or not the system has a unique solution and the number of free parameters in the general solution.

(a) $$\begin{aligned} 2x_1 + 3x_2 &= 3 \\ 4x_1 + 5x_2 &= 5 \end{aligned}$$

(b) $$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 3 \\ 2x_1 + x_2 - x_3 &= 1 \\ 6x_1 + 5x_2 + 2x_3 &= 5 \end{aligned}$$

(c) $$\begin{aligned} x_1 + 3x_2 + 4x_3 &= 1 \\ 2x_1 + 2x_2 - x_3 &= 1 \\ 4x_1 + 5x_2 + 2x_3 &= 5 \end{aligned}$$

(d) $$\begin{aligned} 2x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 2x_2 - x_3 &= 1 \\ 6x_1 + 4x_2 + x_3 &= 4 \end{aligned}$$

(e) $$\begin{aligned} x_1 - x_2 + x_3 + 2x_4 + x_5 &= -1 \\ -x_1 + 3x_2 + 2x_3 + x_4 + x_5 &= 2 \\ 2x_1 + 5x_3 + 7x_4 + 4x_5 &= -1 \\ -x_1 + 5x_2 + 5x_3 + 4x_4 + 3x_5 &= 3 \end{aligned}$$

(f) $$\begin{aligned} x_1 + x_2 + x_3 + 3x_4 &= 2 \\ x_1 + 2x_2 + 3x_3 + 2x_4 &= 4 \\ x_1 + 2x_3 + 2x_4 &= 2 \\ x_1 + 5x_2 + 3x_3 + 3x_4 &= 0 \end{aligned}$$

6) Consider the system of equations

$$\begin{aligned} x_1 + x_2 + 2x_3 &= q \\ x_2 + x_3 + 2x_4 &= 0 \\ x_1 + x_2 + 3x_3 + 3x_4 &= 0 \\ 2x_2 + 5x_3 + px_4 &= 3 \end{aligned}$$

For which values of p and q does the system have

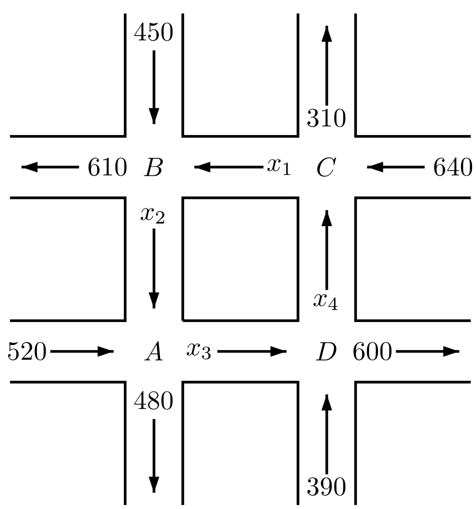
(i) no solutions
(ii) a unique solution
(iii) exactly two solutions
(iv) more than two solutions?

7) In the process of photsynthesis plants use energy from sunlight to convert carbon dioxide and water into glucose and oxygen. The chemical equation of the reaction is of the form

$$x_1 CO_2 + x_2 H_2O \rightarrow x_3 O_2 + x_4 C_6H_{12}O_6$$

Determine the possible values of $x_1,\ x_2,\ x_3$ and x_4.

8) The average hourly volume of traffic in a set of one way streets is given in the figure



Determine $x_1,\ x_2,\ x_3$ and x_4.

9) The following table gives the number of milligrams of vitamins A, B, C contained in one gram of each of the foods $F_1,\ F_2,\ F_3,\ F_4$. A mixture is to be prepared containing precisely 14 mg. of A, 29 mg. of B and 23 mg. of C. Find the greatest amount of F_2 that can be used in the mixture.

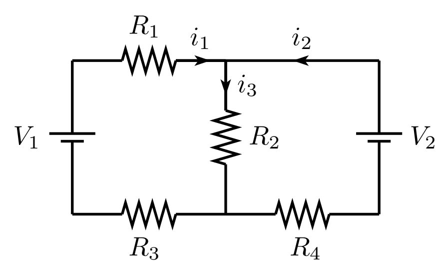
	F_1	F_2	F_3	F_4
A	1	1	1	1
B	1	3	2	1
C	4	0	1	1

10) State whether each of the following statments is true or false. In each case give a *brief* reason.

a) If A is an arbitrary matrix such that the system of equations $A\vec{x} = \vec{b}$ has a unique solution for $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then the system has a unique solution for any three component column vector $\vec{b}$.

b) Any system of 47 homogeneous equations in 19 unknowns whose coefficient matrix has rank greater than or equal to 8 always has at least 11 independent solutions.

11) Find the current through the resistor R_2 in the electrical network below. All of the resistors are 10 ohms and both the voltages are 5 volts.



12) Two weights with masses $m_1 = 1$ gm and $m_2 = 3$ gm are strung out between three springs, attached to floor and ceiling, with spring constants $k_1 = g$ dynes/cm, $k_2 = 2g$ dynes/cm and $k_3 = 3g$ dynes/cm. The gravitational constant is $g = 980$ cm/sec^2. Each weight is a cube of volume 1 cm^3.

a) Suppose that the natural lengths of the springs are $\ell_1 = 10$ cm, $\ell_2 = 15$ cm and $\ell_3 = 15$ cm and that the distance between floor and ceiling is 50 cm. Determine the equilibrium positions of the weights.

b) The bottom weight is pulled down and held 1 cm from its equilibrium position. Find the displacement of the top weight.

Solutions

1) For each of the following systems of linear equations, find the general solution in parametric form. Give a geometric interpretation.

(a) $\begin{aligned} x + 4y &= 5 \\ 2x + 7y &= 6 \end{aligned}$ (b) $\begin{aligned} 2x + 4y &= -12 \\ 6x + 12y &= -24 \end{aligned}$ (c) $\begin{aligned} x + 3y &= 3 \\ 2x + 6y &= 6 \end{aligned}$

(d) $\begin{aligned} 3x + 2y &= 1 \\ 4x + 5y &= 6 \end{aligned}$

Solution. a) Mutiplying equation (1) by 2 and subtracting equation (2) gives $(2x+8y)-(2x+7y) = 10-6$ or $\boxed{y = 4}$. Substituting this into equation (2) gives $2x+28 = 6$ or $\boxed{x = -11}$. In this problem, $x+4y = 5$ is the equation of a line and $2x+7y = 6$ is the equation of a second line. The two lines cross at $(-11, 4)$.

b) Multipying equation (1) by 3 gives $6x + 12y = -36$ which contradicts the requirement from equation (2) that $6x + 12y = -24$. $\boxed{\text{No } (x, y) \text{ satisfies these equations.}}$ In this problem, $2x + 4y = -12$ is the equation of a line and $6x + 12y = -24$ is the equation of a second parallel line that does not intersect the first.

c) Multipying equation (1) by 2 gives $2x + 6y = 6$ which is identical to equation (2). The two equations are equivalent. If we assign any value t to y, then $x = 3 - 3t$ satisfies both equations. The general solution is $\boxed{[x, y] = [3 - 3t, 3t] \text{ for all } t \in \mathbb{R}}$. In this problem, $x + 3y = 6$ is the equation of a line and $2x + 6y = 6$ is a second equation for the same line. The intersection of the two is the same line once again.

d) Multiplying the first equation by 4 and the second by 3 gives

$$12x + 8y = 4 \qquad\qquad 12x + 15y = 18$$

Subtracting the first equation from the second gives $7y = 14$ or $\boxed{y = 2}$. Substituting $y = 2$ into $3x + 2y = 1$ gives $3x + 4 = 1$ or $\boxed{x = -1}$. As in part a, we have been given two lines that intersect a one point, which in this case is $(-1, 2)$.

2) For each of the following systems of linear equations, find the general solution in parametric form. Give a geometric interpretation.

(a) $\begin{aligned} x + 3y + \ \ 4z &= 2 \\ 3x + 8y + 12z &= 5 \end{aligned}$

(b) $\begin{aligned} 3x + 2y - \ \ z &= -2 \\ 6x + 4y - 2z &= \ \ 5 \end{aligned}$

(c) $\begin{aligned} x + \ \ 2y + 3z &= 2 \\ 4x + 10y + 6z &= 4 \end{aligned}$

(d) $\begin{aligned} 2x + 3y + \ \ z &= 0 \\ 4x + 6y + 3z &= 8 \end{aligned}$

Solution. a) Multiplying equation (1) by 3 gives $3x + 9y + 12z = 6$. Subtracting equation (2) from this gives $y = 1$. Substituting $y = 1$ into $x + 3y + 4z = 2$ gives $x + 4z = -1$. We are free to assign z any value t at all. Once we have done so, $x + 4z = -1$ fixes $x = -1 - 4t$. The general solution is $\boxed{[x, y, z] = [-1 - 4t, 1, t] \text{ for all } t \in \mathbb{R}}$. To check this, substitute it into the original equations:

$$\begin{aligned} x + 3y + \ \ 4z &= \ \ (-1 - 4t) + 3 + 4t \ \ = 2 \\ 3x + 8y + 12z &= 3(-1 - 4t) + 8 + 12t = 5 \end{aligned}$$

for all t as desired. In this problem $x + 3y + 4z = 2$ is the equation of a plane and $3x + 8y + 12z = 5$ is the equation of a second plane. The two planes intersect in a line, which passes through $(-1, 1, 0)$ (the solution when $t = 0$) and which has direction vector $[-4, 0, 1]$ (the coefficient of t).

b) Multiplying equation (1) by 2 gives $6x + 4y - 2z = -4$, which contradicts the second equation. $\boxed{\text{No } [x, y, z] \text{ satisfies both equations.}}$ In this problem the two equations represent two planes that are parallel and do not intersect.

c) Multiplying equation (1) by 4 gives $4x + 8y + 12z = 8$. Subtracting equation (2) from this gives $-2y + 6z = 4$. Hence the original system of equations is equivalent to

$$\begin{aligned} x + 2y + 3z &= 2 \\ -2y + 6z &= 4 \end{aligned}$$

Assign z any value t at all. With this value of z, $-2y + 6z = 4$ fixes $y = 3t - 2$ and $x + 2y + 3z = 2$ forces $x + 2(3t - 2) + 3t = 2$ or $x = 6 - 9t$. The general solution is $\boxed{[x, y, z] = [6 - 9t, -2 + 3t, t] \text{ for all } t \in \mathbb{R}}$. To check this, substitute it into the original equations:

$$\begin{aligned} x + \ \ 2y + 3z &= \ \ (6 - 9t) + \ \ 2(-2 + 3t) + 3t = 2 \\ 4x + 10y + 6z &= 4(6 - 9t) + 10(-2 + 3t) + 6t = 4 \end{aligned}$$

for all t as desired. In this problem we have been given two planes that intersect in a line, which passes through $(6, -2, 0)$ (the solution when $t = 0$) and which has direction vector $[-9, 3, 1]$ (the coefficient of t).

d) Multiplying equation (1) by 2 gives $4x + 6y + 2z = 0$. Subtracting this from equation (2) gives $z = 8$. Substituting this back into equation (1) gives $2x + 3y = -8$. We may assign y any value, t we like, provided we set $x = -4 - \frac{3}{2}t$. The general solution is $\boxed{[x, y, z] = [-4 - \frac{3}{2}t, t, 8] \text{ for all } t \in \mathbb{R}}$. Once again, we have been given two planes that intersect in a line, this time passing through $(-4, 0, 8)$ with direction vector $[-3/2, 1, 0]$.

3) For each of the following systems of linear equations, find the general solution in parametric form. Give a geometric interpretation.

(a) $\begin{aligned} x \qquad\quad + 4z &= 1 \\ 2x + 2y + 4z &= 2 \\ 2x - 2y + 8z &= 6 \end{aligned}$ (b) $\begin{aligned} x + 2y + 4z &= 1 \\ 2x + 3y + 7z &= 5 \\ 5x + 5y + 15z &= 9 \end{aligned}$ (c) $\begin{aligned} 3x + 4y + 2z &= 1 \\ 6x + 12y + 5z &= 4 \\ 3x + 8y + 3z &= 3 \end{aligned}$

(d) $\begin{aligned} x + y + z &= 2 \\ x + 2y - z &= 4 \\ x + y + 2z &= 4 \end{aligned}$ (e) $\begin{aligned} x + 3y + 3z &= 5 \\ 2x + 6y + 3z &= 7 \\ x + 3y - 3z &= 2 \end{aligned}$

Solution. a) We are given three equations in three unknowns. Note that the unknown y does not appear in the first equation. Furthermore, we can construct a second equation in x and z by adding together the second and third equations, which yields $4x + 12z = 8$. We now have the system

$$\begin{aligned} x + 4z &= 1 \\ 4x + 12z &= 8 \end{aligned}$$

of two equations in two unknowns. We can eliminate the unknown z from these two equations by subtracting three times the first ($3x + 12z = 3$) from the second, which gives $x = 5$. Substituting $x = 5$ into $x + 4z = 1$ gives $z = -1$ and substituting $x = 5,\ z = -1$ into $2x + 2y + 4z = 2$ gives $y = -2$. The solution is $\boxed{[5, -2, -1]}$. To check, sub back into the original equations

$$\begin{aligned} 5 \qquad\qquad + 4(-1) &= 1 \\ 2(5) + 2(-2) + 4(-1) &= 2 \\ 2(5) - 2(-2) + 8(-1) &= 6 \end{aligned}$$

Each of the three given equations specifies a plane. The three planes have a single point of intersection, namely $(5, -2, -1)$.

b) We are given three equations in three unknowns. We can eliminate the unknown x from equation (2) by subtracting from it 2 times equation (1) and we can eliminate the unknown x from equation (3) by subtracting from it 5 times equation (1):

$$\begin{aligned} (2) - 2(1): \quad -y - z &= 3 \\ (3) - 5(1): \quad -5y - 5z &= 4 \end{aligned}$$

It is impossible to satisfy these two equations because 5 times the first is $-5y - 5z = 15$, which flatly contradicts the second. $\boxed{\text{No } (x, y, z) \text{ satisfies all three given equations}}$. The three planes do not have any point in common.

c) We are given three equations in three unknowns. We can eliminate the unknown x from equation (2) by subtracting from it 2 times equation (1) and we can eliminate the unknown x from equation (3) by subtracting from it equation (1):

$$\begin{aligned} (2) - 2(1): \quad 4y + z &= 2 \\ (3) - (1): \quad 4y + z &= 2 \end{aligned}$$

Any $z = t$ satisfies both of these equations provided we take $y = \frac{1}{2} - \frac{1}{4}t$. Subbing these values of y and z into the first equation gives $3x + (2 - t) + 2t = 1$ or $x = -\frac{1}{3} - \frac{1}{3}t$. The general solution is $\boxed{[x, y, x] = [-\frac{1}{3} - \frac{1}{3}t, \frac{1}{2} - \frac{1}{4}t, t] \text{ for all } t \in \mathbb{R}}$. The three planes intesect in a line. Check:

$$\begin{aligned} 3\big(-\tfrac{1}{3} - \tfrac{1}{3}t\big) + 4\big(\tfrac{1}{2} - \tfrac{1}{4}t\big) + 2t &= 1 \\ 6\big(-\tfrac{1}{3} - \tfrac{1}{3}t\big) + 12\big(\tfrac{1}{2} - \tfrac{1}{4}t\big) + 5t &= 4 \\ 3\big(-\tfrac{1}{3} - \tfrac{1}{3}t\big) + 8\big(\tfrac{1}{2} - \tfrac{1}{4}t\big) + 3t &= 3 \end{aligned}$$

d) We now switch to the augmented matrix notation of §II.2.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & -1 & 4 \\ 1 & 1 & 2 & 4 \end{array}\right] \qquad \begin{array}{r} (1) \\ (2) - (1) \\ (3) - (1) \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

The three equations now are $x+y+z=2,\ y-2z=2,\ z=2$. Substituting $z=2$ into $y-2z=2$ gives $y=6$ and substituting $y=6,\ z=2$ into $x+y+z=2$ gives $x=-6$. The only solution is $\boxed{x=-6,\ y=6,\ z=2}$. To check, substitute back into the original equations:

$$\begin{aligned} x+\ \ y+\ \ z&=-6+\ \ 6+2=2\\ x+2y-\ \ z&=-6+12-2=4\\ x+\ \ y+2z&=-6+\ \ 6+4=4 \end{aligned}$$

e)

$$\left[\begin{array}{ccc|c} 1 & 3 & 3 & 5\\ 2 & 6 & 3 & 7\\ 1 & 3 & -3 & 2 \end{array}\right] \qquad \begin{array}{r} (1)\\ (2)-2(1)\\ (3)-(1) \end{array}\left[\begin{array}{ccc|c} 1 & 3 & 3 & 5\\ 0 & 0 & -3 & -3\\ 0 & 0 & -6 & -3 \end{array}\right]$$

The last two equations, $-6z=-3$ and $-3z=-3$, cannot be simultaneously satisfied. There is $\boxed{\text{no solution}}$.

4) Find the general solution for each of the following systems of linear equations.

(a)
$$\begin{aligned} x_1+3x_2+x_3+\ \ x_4&=1\\ -2x_1-5x_2-x_3-5x_4&=1 \end{aligned}$$

(b)
$$\begin{aligned} x_1+3x_2+2x_3+\ \ x_4&=1\\ x_1+3x_2+\ \ x_3+2x_4&=1\\ x_1+4x_2+\ \ x_3+3x_4&=2 \end{aligned}$$

(c)
$$\begin{aligned} x_1+3x_2+\ \ x_3+\ \ 7x_4&=\ \ 5\\ 3x_1+9x_2+\ \ x_3+11x_4&=17\\ 2x_1+6x_2+2x_3+14x_4&=10 \end{aligned}$$

(d)
$$\begin{aligned} x_1+\ \ 5x_2+3x_3+2x_4+3x_5&=1\\ 2x_1+10x_2+6x_3+5x_4+5x_5&=4 \end{aligned}$$

(e)
$$\begin{aligned} 2x_1+4x_2+2x_3+\ \ 2x_4+\ \ x_5&=18\\ 4x_1+8x_2+6x_3\qquad\qquad+3x_5&=44\\ -2x_1-4x_2+2x_3-10x_4+2x_5&=\ \ 2 \end{aligned}$$

(f)
$$\begin{aligned} x_1+x_2+x_3+x_4&=\ \ 14\\ x_1+x_2-x_3-x_4&=-4\\ x_1-x_2+x_3-x_4&=-2\\ x_1-x_2-x_3+x_4&=\ \ \ 0 \end{aligned}$$

(g)
$$\begin{aligned} x_1\qquad\ \ +\ \ x_3+3x_4&=2\\ x_1+2x_2+3x_3+2x_4&=4\\ 3x_1+2x_2+7x_3\qquad\ &=1\\ x_1\qquad\ \ +3x_3-5x_4&=8 \end{aligned}$$

(h)
$$\begin{aligned} x_1+2x_2+\ \ x_3+2x_4+3x_5&=0\\ 2x_1+5x_2+2x_3+5x_4+7x_5&=1\\ x_1+3x_2+\ \ x_3+3x_4+4x_5&=1\\ x_1+\ \ x_2+\ \ x_3+\ \ x_4+\ \ x_5&=0 \end{aligned}$$

Solution. a)

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 1\\ -2 & -5 & -1 & -5 & 1 \end{array}\right] \qquad \begin{array}{r} (1)\\ (2)+2(1) \end{array}\left[\begin{array}{cccc|c} 1 & 3 & 1 & 1 & 1\\ 0 & 1 & 1 & -3 & 3 \end{array}\right]$$

We now backsolve. We may assign $x_4=s,\ x_3=t$ with $s,\ t$ arbitrary. Then

$$\begin{aligned} x_2+t-3s&=3 &&\Longrightarrow\quad x_2=\ \ 3+\ \ 3s-\ \ t\\ x_1+3(3+3s-t)+t+\ \ s&=1 &&\Longrightarrow\quad x_1=-8-10s+2t \end{aligned}$$

The general solution is $\boxed{x_1=-8-10s+2t,\ x_2=3+3s-t,\ x_3=t,\ x_4=s \text{ for all } s,t\in\mathbb{R}}$. To check, substitute back into the original equations:

$$\begin{aligned} x_1+3x_2+x_3+\ \ x_4&=\qquad(-8-10s+2t)+3(3+3s-t)+t+\ \ s=1\\ -2x_1-5x_2-x_3-5x_4&=-2(-8-10s+2t)-5(3+3s-t)-t-5s=1 \end{aligned}$$

b)

$$\left[\begin{array}{cccc|c} 1 & 3 & 2 & 1 & 1\\ 1 & 3 & 1 & 2 & 1\\ 1 & 4 & 1 & 3 & 2 \end{array}\right] \qquad \begin{array}{r} (1)\\ (2)-(1)\\ (3)-(1) \end{array}\left[\begin{array}{cccc|c} 1 & 3 & 2 & 1 & 1\\ 0 & 0 & -1 & 1 & 0\\ 0 & 1 & -1 & 2 & 1 \end{array}\right] \qquad \begin{array}{r} (1)\\ (3)\\ (2) \end{array}\left[\begin{array}{cccc|c} 1 & 3 & 2 & 1 & 1\\ 0 & 1 & -1 & 2 & 1\\ 0 & 0 & -1 & 1 & 0 \end{array}\right]$$

We now backsolve. We may assign $x_4 = t$ with t arbitrary. Then

$$\begin{aligned} -x_3 + t = 0 &\quad \Longrightarrow \quad x_3 = t \\ x_2 - t + 2t = 1 &\quad \Longrightarrow \quad x_2 = 1 - t \\ x_1 + 3(1-t) + 2t + t = 1 &\quad \Longrightarrow \quad x_1 = -2 \end{aligned}$$

The general solution is $\boxed{x_1 = -2,\ x_2 = 1 - t,\ x_3 = t,\ x_4 = t \text{ for all } t \in \mathbb{R}}$.

c)

$$\left[\begin{array}{cccc|c} 1 & 3 & 1 & 7 & 5 \\ 3 & 9 & 1 & 11 & 17 \\ 2 & 6 & 2 & 14 & 10 \end{array}\right] \qquad \begin{array}{r} (1) \\ (2)-3(1) \\ (3)-2(1) \end{array} \left[\begin{array}{cccc|c} 1 & 3 & 1 & 7 & 5 \\ 0 & 0 & -2 & -10 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

We now backsolve. We may assign $x_2 = t,\ x_4 = s$ with s, t arbitrary. Then

$$\begin{aligned} -2x_3 - 10s = 2 &\quad \Longrightarrow \quad x_3 = -1 - 5s \\ x_1 + 3t + (-1 - 5s) + 7s = 5 &\quad \Longrightarrow \quad x_1 = 6 - 2s - 3t \end{aligned}$$

The general solution is $\boxed{x_1 = 6 - 2s - 3t,\ x_2 = t,\ x_3 = -1 - 5s,\ x_4 = s \text{ for all } s, t \in \mathbb{R}}$.

d)

$$\left[\begin{array}{ccccc|c} 1 & 5 & 3 & 2 & 3 & 1 \\ 2 & 10 & 6 & 5 & 5 & 4 \end{array}\right] \qquad \begin{array}{r} (1) \\ (2)-2(1) \end{array} \left[\begin{array}{ccccc|c} 1 & 5 & 3 & 2 & 3 & 1 \\ 0 & 0 & 0 & 1 & -1 & 2 \end{array}\right]$$

We now backsolve. We may assign $x_5 = s$ with s arbitrary. Then, the last equation

$$x_4 - x_5 = 2 \qquad \Longrightarrow \qquad x_4 = 2 + s$$

Now, we may assign $x_3 = t,\ x_2 = u$ with $t,\ u$ arbitrary. Then, the first equation

$$x_1 + 5u + 3t + 2(2 + s) + 3s = 1 \qquad \Longrightarrow \qquad x_1 = -3 - 5s - 3t - 5u$$

All together $\boxed{x_1 = -3 - 5s - 3t - 5u,\ x_2 = u,\ x_3 = t,\ x_4 = 2 + s,\ x_5 = s}$.

e)

$$\left[\begin{array}{ccccc|c} 2 & 4 & 2 & 2 & 1 & 18 \\ 4 & 8 & 6 & 0 & 3 & 44 \\ -2 & -4 & 2 & -10 & 2 & 2 \end{array}\right] \qquad \begin{array}{r} (1) \\ (2)-2(1) \\ (3)+(1) \end{array} \left[\begin{array}{ccccc|c} 2 & 4 & 2 & 2 & 1 & 18 \\ 0 & 0 & 2 & -4 & 1 & 8 \\ 0 & 0 & 4 & -8 & 3 & 20 \end{array}\right]$$

$$\begin{array}{r} (1) \\ (2) \\ (3)-2(2) \end{array} \left[\begin{array}{ccccc|c} 2 & 4 & 2 & 2 & 1 & 18 \\ 0 & 0 & 2 & -4 & 1 & 8 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

We now backsolve. The last equation forces $x_5 = 4$. Substituting this into the middle equation gives $2x_3 - 4x_4 + 4 = 8$. So we may assign $x_4 = s$, arbitrary and then $x_3 = 2 + 2s$. Subbing, the now known values of $x_3,\ x_4,\ x_5$ into the first equation gives $2x_1 + 4x_2 + 2(2 + 2s) + 2s + 4 = 18$. So we may assign $x_2 = t$, arbitrary, and then $x_1 = 5 - 3s - 2t$. All together $\boxed{x_1 = 5 - 3s - 2t,\ x_2 = t,\ x_3 = 2 + 2s,\ x_4 = s,\ x_5 = 4}$.

f)

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 14 \\ 1 & 1 & -1 & -1 & -4 \\ 1 & -1 & 1 & -1 & -2 \\ 1 & -1 & -1 & 1 & 0 \end{array}\right] \qquad \begin{array}{r} (1) \\ (2)-(1) \\ (3)-(1) \\ (4)-(1) \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 14 \\ 0 & 0 & -2 & -2 & -18 \\ 0 & -2 & 0 & -2 & -16 \\ 0 & -2 & -2 & 0 & -14 \end{array}\right]$$

$$\begin{array}{r} (1) \\ (3) \\ (2) \\ (4) \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 14 \\ 0 & -2 & 0 & -2 & -16 \\ 0 & 0 & -2 & -2 & -18 \\ 0 & -2 & -2 & 0 & -14 \end{array}\right] \qquad \begin{array}{r} (1) \\ (2) \\ (3) \\ (4)-(2) \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 14 \\ 0 & -2 & 0 & -2 & -16 \\ 0 & 0 & -2 & -2 & -18 \\ 0 & 0 & -2 & 2 & 2 \end{array}\right]$$

$$\begin{array}{r} (1) \\ (2) \\ (3) \\ (4)-(3) \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 14 \\ 0 & -2 & 0 & -2 & -16 \\ 0 & 0 & -2 & -2 & -18 \\ 0 & 0 & 0 & 4 & 20 \end{array}\right]$$

We now backsolve. The last equation, $4x_4 = 20$, forces $x_4 = 5$. Then the third equation, $-2x_3 - 2x_4 = -18$ or $-2x_3 - 10 = -18$ forces $x_3 = 4$. Then the second equation $-2x_2 - 2x_4 = -16$ or $-2x_2 - 10 = -16$ forces $x_2 = 3$. Finally, the first equation $x_1 + x_2 + x_3 + x_4 = 14$ or $x_1 + 3 + 4 + 5 = 14$ forces $x_1 = 2$. All together $\boxed{x_1 = 2,\ x_2 = 3,\ x_3 = 4,\ x_4 = 5}$.

g)

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 3 & 2 \\ 1 & 2 & 3 & 2 & 4 \\ 3 & 2 & 7 & 0 & 1 \\ 1 & 0 & 3 & -5 & 8 \end{array}\right] \qquad \begin{array}{c} (1) \\ (2)-(1) \\ (3)-3(1) \\ (4)-(1) \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 3 & 2 \\ 0 & 2 & 2 & -1 & 2 \\ 0 & 2 & 4 & -9 & -5 \\ 0 & 0 & 2 & -8 & 6 \end{array}\right]$$

$$\begin{array}{c} (1) \\ (2) \\ (3)-(2) \\ (4) \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 3 & 2 \\ 0 & 2 & 2 & -1 & 2 \\ 0 & 0 & 2 & -8 & -7 \\ 0 & 0 & 2 & -8 & 6 \end{array}\right]$$

The last two equations cannot be simultaneously satisfied. So there is $\boxed{\text{no solution}}$.

h)

$$\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 3 & 0 \\ 2 & 5 & 2 & 5 & 7 & 1 \\ 1 & 3 & 1 & 3 & 4 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right] \qquad \begin{array}{c} (1) \\ (2)-2(1) \\ (3)-(1) \\ (4)-(1) \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 3 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 & -2 & 0 \end{array}\right]$$

$$\begin{array}{c} (1) \\ (2) \\ (3)-(2) \\ (4)+(2) \end{array} \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 2 & 3 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right]$$

We now backsolve. The last equation, $-x_5 = 1$, forces $x_5 = -1$. The third equation places absolutely no restriction on any variable. The second equation $x_2 + x_4 + x_5 = 1$, or $x_2 + x_4 = 2$, allows us to set $x_4 = s$, arbitrary, and then forces $x_2 = 2 - s$. Finally, the first equation $x_1 + 2x_2 + x_3 + 2x_4 + 3x_5 = 0$ or $x_1 + 2(2 - s) + x_3 + 2s - 3 = 0$, allows us to set $x_3 = t$, arbitrary, and then forces $x_1 = -1 - t$. All together $\boxed{x_1 = -1 - t,\ x_2 = 2 - s,\ x_3 = t,\ x_4 = s,\ x_5 = -1 \text{ for any } s, t \in \mathbb{R}}$.

5) Construct the augmented matrix for each of the following linear systems of equations. In each case determine the rank of the coefficient matrix, the rank of the augmented matrix, whether or not the system has any solutions, whether or not the system has a unique solution and the number of free parameters in the general solution.

(a)
$$\begin{aligned} 2x_1 + 3x_2 &= 3 \\ 4x_1 + 5x_2 &= 5 \end{aligned}$$

(b)
$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 3 \\ 2x_1 + x_2 - x_3 &= 1 \\ 6x_1 + 5x_2 + 2x_3 &= 5 \end{aligned}$$

(c)
$$\begin{aligned} x_1 + 3x_2 + 4x_3 &= 1 \\ 2x_1 + 2x_2 - x_3 &= 1 \\ 4x_1 + 5x_2 + 2x_3 &= 5 \end{aligned}$$

(d)
$$\begin{aligned} 2x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 2x_2 - x_3 &= 1 \\ 6x_1 + 4x_2 + x_3 &= 4 \end{aligned}$$

(e)
$$\begin{aligned} x_1 - x_2 + x_3 + 2x_4 + x_5 &= -1 \\ -x_1 + 3x_2 + 2x_3 + x_4 + x_5 &= 2 \\ 2x_1 \qquad\quad + 5x_3 + 7x_4 + 4x_5 &= -1 \\ -x_1 + 5x_2 + 5x_3 + 4x_4 + 3x_5 &= 3 \end{aligned}$$

(f)
$$\begin{aligned} x_1 + x_2 + x_3 + 3x_4 &= 2 \\ x_1 + 2x_2 + 3x_3 + 2x_4 &= 4 \\ x_1 \qquad\ + 2x_3 + 2x_4 &= 2 \\ x_1 + 5x_2 + 3x_3 + 3x_4 &= 0 \end{aligned}$$

Solution. We use $[A]$ to denote the coefficient matrix and $[A|b]$ to denote the augmented matrix.

a)

$$\left[\begin{array}{cc|c} 2 & 3 & 3 \\ 4 & 5 & 5 \end{array}\right] \qquad \begin{array}{c} (1) \\ (2)-2(1) \end{array} \left[\begin{array}{cc|c} 2 & 3 & 3 \\ 0 & -1 & -1 \end{array}\right]$$

$\boxed{\#\text{unknowns} = \text{rank}\,[A] = \text{rank}\,[A|b] = 2. \text{ The system has a unique solution.}}$

b)

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 3 \\ 2 & 1 & -1 & 1 \\ 6 & 5 & 2 & 5 \end{array}\right] \qquad \begin{array}{r} (1) \\ (2)-(1) \\ (3)-3(1) \end{array} \left[\begin{array}{ccc|c} 2 & 3 & 4 & 3 \\ 0 & -2 & -5 & -2 \\ 0 & -4 & -10 & -4 \end{array}\right]$$

$$\begin{array}{r} (1) \\ (2) \\ (3)-2(2) \end{array} \left[\begin{array}{ccc|c} 2 & 3 & 4 & 3 \\ 0 & -2 & -5 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\boxed{\#\text{unknowns} = 3,\ \text{rank}\,[A] = \text{rank}\,[A|b] = 2. \text{ The system has a one parameter family of solutions.}}$$

c)

$$\left[\begin{array}{ccc|c} 1 & 3 & 4 & 1 \\ 2 & 2 & -1 & 1 \\ 4 & 5 & 2 & 5 \end{array}\right] \qquad \begin{array}{r} (1) \\ (2)-2(1) \\ (3)-4(1) \end{array} \left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 0 & -4 & -9 & -1 \\ 0 & -7 & -14 & 1 \end{array}\right]$$

$$\begin{array}{r} (1) \\ (2) \\ (3)-\frac{7}{4}(2) \end{array} \left[\begin{array}{ccc|c} 2 & 3 & 4 & 1 \\ 0 & -4 & -9 & -1 \\ 0 & 0 & \frac{7}{4} & \frac{11}{4} \end{array}\right]$$

$$\boxed{\#\text{unknowns} = \text{rank}\,[A] = \text{rank}\,[A|b] = 3. \text{ The system has a unique solution.}}$$

d)

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 2 & 2 & -1 & 1 \\ 6 & 4 & 1 & 4 \end{array}\right] \qquad \begin{array}{r} (1) \\ (2)-(1) \\ (3)-3(1) \end{array} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 1 & -2 & -2 \end{array}\right]$$

$$\begin{array}{r} (1) \\ (2) \\ (3)-(2) \end{array} \left[\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

$$\boxed{\text{rank}\,[A] = 2 < \text{rank}\,[A|b] = 3. \text{ The system no solution.}}$$

e)

$$\left[\begin{array}{ccccc|c} 1 & -1 & 1 & 2 & 1 & -1 \\ -1 & 3 & 2 & 1 & 1 & 2 \\ 2 & 0 & 5 & 7 & 4 & -1 \\ -1 & 5 & 5 & 4 & 3 & 3 \end{array}\right] \qquad \begin{array}{r} (1) \\ (2)+(1) \\ (3)-2(1) \\ (4)+(1) \end{array} \left[\begin{array}{ccccc|c} 1 & -1 & 1 & 2 & 1 & -1 \\ 0 & 2 & 3 & 3 & 2 & 1 \\ 0 & 2 & 3 & 3 & 2 & 1 \\ 0 & 4 & 6 & 6 & 4 & 2 \end{array}\right]$$

$$\begin{array}{r} (1) \\ (2) \\ (3)-(2) \\ (4)-2(2) \end{array} \left[\begin{array}{ccccc|c} 1 & -1 & 1 & 2 & 1 & -1 \\ 0 & 2 & 3 & 3 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\boxed{\#\text{unknowns} = 5 > \text{rank}\,[A] = \text{rank}\,[A|b] = 2. \text{ The system has a three parameter family of solutions.}}$$

f)

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 2 \\ 1 & 2 & 3 & 2 & 4 \\ 1 & 0 & 2 & 2 & 2 \\ 1 & 5 & 3 & 3 & 0 \end{array}\right] \qquad \begin{array}{r} (1) \\ (2)-(1) \\ (3)-(1) \\ (4)-(1) \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 2 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & -1 & 1 & -1 & 0 \\ 0 & 4 & 2 & 0 & -2 \end{array}\right]$$

$$\begin{array}{r} (1) \\ (2) \\ (3)+(2) \\ (4)-4(2) \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 2 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 3 & -2 & 2 \\ 0 & 0 & -6 & 4 & -10 \end{array}\right] \qquad \begin{array}{r} (1) \\ (2) \\ (3) \\ (4)+2(3) \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 3 & 2 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 3 & -2 & 2 \\ 0 & 0 & 0 & 0 & -6 \end{array}\right]$$

$$\boxed{\text{rank}\,[A] = 3 < \text{rank}\,[A|b] = 4. \text{ The system no solution.}}$$

6) Consider the system of equations

$$\begin{aligned} x_1 + \ x_2 + 2x_3 \qquad &= q \\ x_2 + \ \ x_3 + 2x_4 &= 0 \\ x_1 + \ x_2 + 3x_3 + 3x_4 &= 0 \\ 2x_2 + 5x_3 + px_4 &= 3 \end{aligned}$$

For which values of p and q does the system have

(i) no solutions
(ii) a unique solution
(iii) exactly two solutions
(iv) more than two solutions?

Solution.

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & q \\ 0 & 1 & 1 & 2 & 0 \\ 1 & 1 & 3 & 3 & 0 \\ 0 & 2 & 5 & p & 3 \end{array}\right] \qquad \begin{array}{c} (1) \\ (2) \\ (3)-(1) \\ (4) \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & q \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 & -q \\ 0 & 2 & 5 & p & 3 \end{array}\right]$$

$$\begin{array}{c} (1) \\ (2) \\ (3) \\ (4)-2(2) \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & q \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 & -q \\ 0 & 0 & 3 & p-4 & 3 \end{array}\right] \qquad \begin{array}{c} (1) \\ (2) \\ (3) \\ (4)-3(3) \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & q \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 & -q \\ 0 & 0 & 0 & p-13 & 3+3q \end{array}\right]$$

The ranks of the coefficient and the augmented matrices are both 4 if $p \neq 13$. If $p = 13$, the rank of the coefficient matrix is 3. In this case, the augmented matrix has rank 3 if $3 + 3q = 0$ and 4 otherwise. There

(i) are no solutions if $p = 13, \ q \neq -1$
(ii) is exactly one solution if $p \neq 13$
(iii) are **NEVER** exactly two solutions for any linear system
(iv) are infinitely many solutions if $p = 13, \ q = -1$

7) In the process of photsynthesis plants use energy from sunlight to convert carbon dioxide and water into glucose and oxygen. The chemical equation of the reaction is of the form

$$x_1 \mathrm{CO}_2 + x_2 \mathrm{H}_2\mathrm{O} \to x_3 \mathrm{O}_2 + x_4 \mathrm{C}_6\mathrm{H}_{12}\mathrm{O}_6$$

Determine the possible values of $x_1, \ x_2, \ x_3$ and x_4.

Solution. The same number of carbon atoms must appear on the left hand side of the chemical equation as on the right. This is the case if and only if

$$x_1 = 6x_4$$

Similarly, the number of oxygen atoms on the two sides must match so that

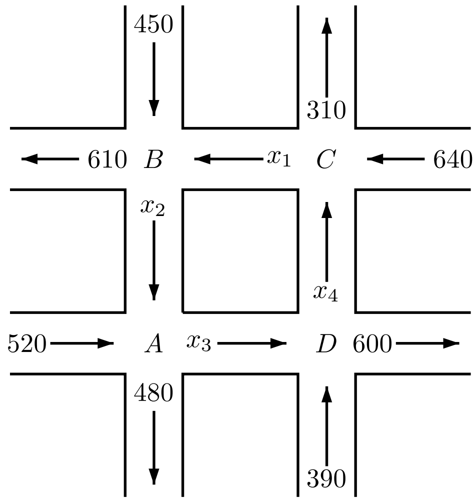
$$2x_1 + x_2 = 2x_3 + 6x_4$$

and the number of hydrogen atoms must match so that

$$2x_2 = 12x_4$$

Substituting the first equation $6x_4 = x_1$ into the third gives $2x_2 = 2x_1$ or $x_1 = x_2$. Substituting $x_2 = x_1$ and $6x_4 = x_1$ into the middle equation gives $2x_1 + x_1 = 2x_3 + x_1$ or $x_1 = x_3$. Since one cannot have a negative or fractional number of molecules, the general solution is $\boxed{x_1 = x_2 = x_3 = 6n, \ x_4 = n, \ n = 0, 1, 2, 3, \cdots}$.

8) The average hourly volume of traffic in a set of one way streets is given in the figure



Determine $x_1,\ x_2,\ x_3$ and x_4.

Solution. Conservation of cars at the four intersections $A,\ B,\ C$ and D imply,

$$
\begin{aligned}
x_2 + 520 &= x_3 + 480 \\
x_1 + 450 &= x_2 + 610 \\
x_4 + 640 &= x_1 + 310 \\
x_3 + 390 &= x_4 + 600
\end{aligned}
$$

respectively. The augmented matrix for this system is

$$
\left[\begin{array}{cccc|c} 0 & 1 & -1 & 0 & -40 \\ 1 & -1 & 0 & 0 & 160 \\ -1 & 0 & 0 & 1 & -330 \\ 0 & 0 & 1 & -1 & 210 \end{array}\right]
\qquad
\begin{array}{c} (2) \\ (1) \\ (3)+(2) \\ (4) \end{array}
\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 160 \\ 0 & 1 & -1 & 0 & -40 \\ 0 & -1 & 0 & 1 & -170 \\ 0 & 0 & 1 & -1 & 210 \end{array}\right]
$$

$$
\begin{array}{c} (1) \\ (2) \\ (3)+(2) \\ (4) \end{array}
\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 160 \\ 0 & 1 & -1 & 0 & -40 \\ 0 & 0 & -1 & 1 & -210 \\ 0 & 0 & 1 & -1 & 210 \end{array}\right]
\qquad
\begin{array}{c} (1) \\ (2) \\ (3) \\ (4)+(3) \end{array}
\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 160 \\ 0 & 1 & -1 & 0 & -40 \\ 0 & 0 & -1 & 1 & -210 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]
$$

We may assign $x_4 = t$ with t arbitrary (except that all the x's must be nonnegative integers). Then

- equation (3) forces $x_3 = t + 210$,
- equation (2) forces $x_2 = -40 + x_3 = t + 170$ and
- equation (1) forces $x_1 = 160 + x_2 = 330 + t$.

The general solution is $\boxed{x_1 = 330 + t,\ x_2 = t + 170,\ x_3 = t + 210,\ x_4 = t,\ t = 0, 1, 2, 3, \cdots}$. The variable t cannot be determined from the given data. It can be thought of as resulting from a stream of cars driving around the loop $ADCBA$.

9) The following table gives the number of milligrams of vitamins A, B, C contained in one gram of each of the foods $F_1,\ F_2,\ F_3,\ F_4$. A mixture is to be prepared containing precisely 14 mg. of A, 29 mg. of B and 23 mg. of C. Find the greatest amount of F_2 that can be used in the mixture.

	F_1	F_2	F_3	F_4
A	1	1	1	1
B	1	3	2	1
C	4	0	1	1

Solution. Denote by x_i the amount of F_i in the mixture. The problem requires

$$\begin{aligned} x_1 + \ \ x_2 + \ \ x_3 + x_4 &= 14 \\ x_1 + 3x_2 + 2x_3 + x_4 &= 29 \\ 4x_1 \qquad\quad + \ \ x_3 + x_4 &= 23 \end{aligned}$$

Switching to augmented matrix notation

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 14 \\ 1 & 3 & 2 & 1 & 29 \\ 4 & 0 & 1 & 1 & 23 \end{array}\right] \quad \begin{array}{c} (1) \\ (2)-(1) \\ (3)-4(1) \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 14 \\ 0 & 2 & 1 & 0 & 15 \\ 0 & -4 & -3 & -3 & -33 \end{array}\right] \quad \begin{array}{c} (1) \\ (2) \\ (3)+2(2) \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 14 \\ 0 & 2 & 1 & 0 & 15 \\ 0 & 0 & -1 & -3 & -3 \end{array}\right]$$

The general solution is $x_4 = t,\ x_3 = 3-3t,\ x_2 = (15-3+3t)/2 = 6+3t/2,\ x_1 = 14-(6+3t/2)-(3-3t)-t = 5+t/2$. As x_3 must remain nonegative, $t \le 1$. Consequently, at most $\boxed{\text{7.5 mg.}}$ of F_2 may be used.

10) State whether each of the following statments is true or false. In each case give a *brief* reason.

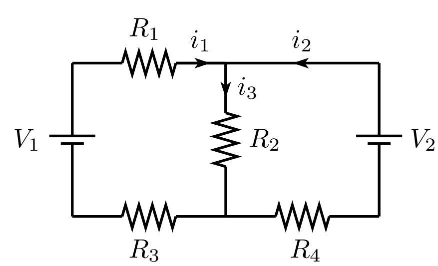
a) If A is an arbitrary matrix such that the system of equations $A\vec{x} = \vec{b}$ has a unique solution for $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then the system has a unique solution for any three component column vector $\vec{b}$.

b) Any system of 47 homogeneous equations in 19 unknowns whose coefficient matrix has rank greater than or equal to 8 always has at least 11 independent solutions.

Solution. a) This statement is $\boxed{\text{true}}$. The system $A\vec{x} = \vec{b}$ has a unique solution for the given $\vec{b}$ if and only if both the coefficient matrix A and augmented matrix $[A|\vec{b}]$ have rank 3. But then both the coefficient matrix A and the augmented matrix $[A|\vec{b}]$ have rank 3 for any $\vec{b}$.

b) This statement is $\boxed{\text{false}}$. The coefficient matrix could have rank as large as 19. In this event the system has a unique solution, namely $\vec{x} = 0$.

11) Find the current through the resistor R_2 in the electrical network below. All of the resistors are 10 ohms and both the voltages are 5 volts.



Solution. Let the currents $i_1,\ i_2$ and i_3 be as in the figure. Conservation of current at the top node forces

$$i_1 + i_2 = i_3$$

The voltage across the battery V_1 has to be same as the voltage through R_1, R_2 and R_3 and the voltage across the battery V_2 has to be same as the voltage through R_2 and R_4. In equations

$$\begin{aligned} R_1 i_1 + R_2 i_3 + R_3 i_1 &= V_1 \\ R_2 i_3 + R_4 i_2 &= V_2 \end{aligned}$$

Substituting in the values of the R_j's and V_j's and cleaning up the equations

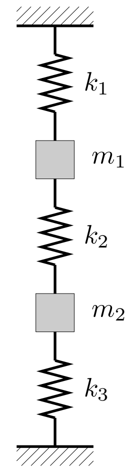
$$\begin{aligned} 20i_1 \qquad + 10i_3 &= 5 \\ 10i_2 + 10i_3 &= 5 \\ i_1 + \ \ i_2 - \ \ \ i_3 &= 0 \end{aligned}$$

Subbing $i_1 = -i_2 + i_3$ into the first equation

$$\begin{aligned} -20i_2 + 30i_3 &= 5 \\ 10i_2 + 10i_3 &= 5 \end{aligned}$$

Adding twice the second equation to the first gives $50i_3 = 15$ and hence $\boxed{0.3 \text{ amps}}$. For the record, $i_1 = 0.1$ and $i_2 = 0.2$.

12) Two weights with masses $m_1 = 1$ gm and $m_2 = 3$ gm are strung out between three springs, attached to floor and ceiling, with spring constants $k_1 = g$ dynes/cm, $k_2 = 2g$ dynes/cm and $k_3 = 3g$ dynes/cm. The gravitational constant is $g = 980$ cm/sec^2. Each weight is a cube of volume 1 cm^3.



a) Suppose that the natural lengths of the springs are $\ell_1 = 10$ cm, $\ell_2 = 15$ cm and $\ell_3 = 15$ cm and that the distance between floor and ceiling is 50 cm. Determine the equilibrium positions of the weights.

b) The bottom weight is pulled down and held 1 cm from its equilibrium position. Find the displacement of the top weight.

Solution. a) Denote by x_1 and x_2 the distances from the centres of the two weights to the floor. Then the lengths of the three springs are $50-x_1-.5$, x_1-x_2-1 and $x_2-.5$ respectively. So the forces exerted by the three springs are $(50-x_1-.5-\ell_1)g$, $2(x_1-x_2-1-\ell_2)g$ and $3(x_2-.5-\ell_3)g$ respectively, with a positive force signifying that the spring is trying to pull its ends closer together. So m_1 has a force of $(50-x_1-.5-\ell_1)g$ acting upward on it and forces of $2(x_1-x_2-1-\ell_2)g$ and m_1g (gravity) acting downward on it. In equilibrium these forces must balance. Similarly m_2 is subject to an upward force of $2(x_1-x_2-1-\ell_2)g$ and downward forces of $3(x_2-.5-\ell_3)g$ and m_2g which must also balance in equilibrium. The equations encoding force balance in equilibrium are

$$\begin{aligned} (50-x_1-.5-\ell_1)g &= 2(x_1-x_2-1-\ell_2)g + m_1g \\ 2(x_1-x_2-1-\ell_2)g &= 3(x_2-.5-\ell_3)g + m_2g \end{aligned}$$

Substituting in the given values of $m_1,\ m_2,\ \ell_1,\ \ell_2$ and ℓ_3

$$\begin{aligned} (39.5-x_1)g &= 2(x_1-x_2-16)g + g \\ 2(x_1-x_2-16)g &= 3(x_2-15.5)g + 3g \end{aligned}$$

and simplifying

$$\begin{aligned} 3x_1 - 2x_2 &= 70.5 \\ 2x_1 - 5x_2 &= -11.5 \end{aligned}$$

Multiplying the first equation by 2 and subtracting from it three times the second gives $11x_2 = 2\times 70.5 + 34.5 = 175.5$ or $\boxed{x_2 \approx 15.95}$. Subbing back into the second equation gives $x_1 = (5\times 175.5/11 - 11.5)/2$ or $\boxed{x_1 \approx 34.14}$.

b) The force balance equation for m_1, $3x_1 - 2x_2 = 70.5$ determines x_1 in terms of x_2 through

$$x_1 = \tfrac{1}{3}(2x_2 + 70.5)$$

If x_2 is decreased by 1, $\boxed{x_1 \text{ is decreased by } 2/3}$.