1) Evaluate
\[ 1 + 2x + 3x^2 + 4x^3 + \cdots + 100x^{99} \]

b) \( \sum_{j=1}^{n} \frac{j}{2^j} \)

Solution. a)

\[ 1 + 2x + 3x^2 + 4x^3 + \cdots + 100x^{99} = \sum_{i=1}^{100} ix^{i-1} = \frac{d}{dx} \sum_{i=0}^{100} x^i = \frac{d}{dx} \frac{1-x^{101}}{1-x} \]

\[ = \frac{-101x^{100}(1-x)+(1-x^{101})}{(1-x)^2} = \frac{1-101x^{100}+100x^{101}}{(1-x)^2} \]

The right hand side is not defined for \( x = 1 \), but this case may be handled by using l'Hôpital’s rule to take the limit \( x \to 1 \):

\[ 1 + 2x + 3x^2 + 4x^3 + \cdots + 100x^{99} \bigg|_{x=1} = \lim_{x \to 1} 1 + 2x + 3x^2 + 4x^3 + \cdots + 100x^{99} \]

\[ = \lim_{x \to 1} \frac{1-101x^{100}+100x^{101}}{(1-x)^2} \]

\[ = \lim_{x \to 1} \frac{-10100x^{99}+10100x^{100}}{2(x-1)} \]

\[ = \lim_{x \to 1} \frac{-999990x^{98}+990000x^{99}}{2} = 5050 \]

You can also use \( 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \) to handle this case.

b) Let \( x = \frac{1}{2} \).

\[ \sum_{j=1}^{n} \frac{j}{2^j} = \sum_{j=1}^{n} jx^j = x \sum_{j=1}^{n} jx^{j-1} = x \frac{d}{dx} \sum_{j=0}^{n} x^j = x \frac{d}{dx} \frac{1-x^{n+1}}{1-x} \]

\[ = x \cdot \frac{-(n+1)x^n(1-x)+(1-x^{n+1})}{(1-x)^2} \]

Subbing \( x = \frac{1}{2} \)

\[ \sum_{j=1}^{n} \frac{j}{2^j} = \frac{1}{2} \cdot \frac{-(n+1)(1/2)^{n+1}+(1-(1/2)^{n+1})}{(1/2)^2} = 2 \left[ 1 - (n + 2) \left( \frac{1}{2} \right)^{n+1} \right] = 2 - \frac{n+2}{2^n} \]
2) Find the area of a circle as a limit of the areas of inscribed polygons.

**Solution.** Inscribe a polygon with \( n \) equal sides in a circle of radius \( r \). The figure at the right illustrates such a polygon with \( n = 6 \). The polygon is built up out of \( n \) equal triangles. Each triangle has one vertex, with angle \( \frac{1}{n}2\pi \), at the centre of the circle. Drop a perpendicular from that vertex to the centre of the opposite side of the triangle. This perpendicular bisects the angle we were just talking about. The length of the perpendicular is \( r \cos \frac{\pi}{n} \). We can also view this as the height of the triangle. The base of the triangle \( \frac{\pi}{n} \) has length \( 2r \sin \frac{\pi}{n} \). So one triangle has area

\[
\frac{1}{2} \text{(base)} \times \text{(height)} = \frac{1}{2} \left( r \cos \frac{\pi}{n} \right) \left( 2r \sin \frac{\pi}{n} \right) = \frac{1}{2} r^2 \left( 2 \sin \frac{\pi}{n} \cos \frac{\pi}{n} \right) = \frac{1}{2} r^2 \sin \frac{2\pi}{n}
\]

The area of the polygon is the area of \( n \) triangles or \( \frac{n}{2} r^2 \sin \frac{2\pi}{n} \). The area of the circle is

\[
\lim_{n \to \infty} \frac{n}{2} r^2 \sin \frac{2\pi}{n} = \lim_{x \to 0} \frac{\pi r^2 \sin x}{x} = \pi r^2 \lim_{x \to 0} \frac{\sin x}{x}
\]

where we have substituted \( x = \frac{2\pi}{n} \), or equivalently \( n = \frac{2\pi}{x} \). By L'Hôpital’s rule,

\[
\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1,
\]

so the area of the circle is \( \pi r^2 \).

3) Evaluate the following integrals by interpreting it as an area.

a) \( \int_{-2}^{2} \sqrt{4-x^2} \, dx \)

b) \( \int_{0}^{3} |3x-5| \, dx \)

c) \( \int_{0}^{1} \sqrt{4-x^2} \, dx \)

**Solution.** Recall that the area between the \( x \)-axis and the curve \( y = f(x) \) with \( x \) running from \( x = a \) to \( x = b \) is \( \int_{a}^{b} f(x) \, dx \).

a) The integral \( \int_{-2}^{2} \sqrt{4-x^2} \, dx \) represents the area between the \( x \)-axis and the curve \( y = \sqrt{4-x^2} \) with \( x \) running from \(-2\) to \( 2 \). As the curve \( y = \sqrt{4-x^2} \) is the top half of the curve \( y^2 = 4-x^2 \) or \( x^2 + y^2 = 4 \), and \( x = \pm 2 \) are the two values of \( x \) for which \( y = 0 \), the integral \( \int_{-2}^{2} \sqrt{4-x^2} \, dx \) represents the area of the top half of the interior of the circle \( x^2 + y^2 = 4 \). This circle has radius \( 2 \) and hence area \( \pi 2^2 = 4\pi \). The integral is half of this, or \( \frac{1}{2} \pi 2^2 \).

b) Observe that \( 3x-5 \) takes the value zero for \( x = \frac{5}{3} \), is positive for \( x > \frac{5}{3} \) and negative for \( x < \frac{5}{3} \). Hence

\[
|3x-5| = \begin{cases} 
3x-5 & \text{if } x > \frac{5}{3} \\
-(3x-5) & \text{if } x < \frac{5}{3}
\end{cases}
\]

and

\[
\int_{0}^{3} |3x-5| \, dx = \int_{0}^{5/3} |3x-5| \, dx + \int_{5/3}^{3} |3x-5| \, dx = \int_{0}^{5/3} -(3x-5) \, dx + \int_{5/3}^{3} (3x-5) \, dx
\]


The first integral represents the area between the $x$–axis and the straight line $y = 5 – 3x$ with $x$ running from 0 (where $y = 5$) to $\frac{5}{3}$ (where $y = 0$). This is a triangle with height 5, base $\frac{5}{3}$ and area $\frac{1}{2} \cdot \frac{5}{3} \cdot 5 = \frac{25}{6}$. The second integral represents the area between the $x$–axis and the straight line $y = 3x – 5$ with $x$ running from $\frac{5}{3}$ (where $y = 0$) to $x = 3$ (where $y = 4$). This is a triangle with height 4, base $3 – \frac{5}{3} = \frac{4}{3}$ and area $\frac{1}{2} \cdot \frac{4}{3} \cdot 4 = \frac{16}{6}$. All together

$$\int_0^3 |3x - 5| \, dx = \int_0^{5/3} -(3x - 5) \, dx + \int_{5/3}^3 (3x - 5) \, dx = \frac{25}{6} + \frac{16}{6} = \frac{41}{6}$$

c) The integral $\int_0^1 \sqrt{4 - x^2} \, dx$ represents the area between the $x$–axis and the top half of the circle $x^2 + y^2 = 4$ with $0 \leq x \leq 1$. This region consists of the circular sector $OAB$, which is the fraction $\frac{\pi/6}{2\pi} = \frac{1}{12}$ of a full disk of radius 2, and the triangle $OBC$, which has base 1 and height $\sqrt{3}$. The total area is thus

$$\frac{1}{12} \pi 2^2 + \frac{1}{2} \times 1 \times \sqrt{3} = \frac{\pi}{3} + \frac{\sqrt{3}}{2}.$$ 

4) Let $h(x) = \int_{-5}^{\sin x} t \cos (t^3) \, dt$. Evaluate $h'(x)$.

**Solution.** Define

$$f(u) = \int_u^{-5} t \cos (t^3) \, dt \quad \text{and} \quad g(x) = \sin x$$

Then

$$\frac{df}{du} = u \cos (u^3) \quad \text{and} \quad \frac{dg}{dx} = \cos x$$

and $h(x) = f(g(x))$ so that, by the chain rule,

$$h'(x) = f'(g(x))g'(x) = u \cos (u^3) \Big|_{u=g(x)} \cos x = \sin x \cos (\sin^3 x) \cos x$$

5) Find the area of the region bounded by the parabola $y = x^2$, the tangent line to this parabola at $(1, 1)$ and the $x$–axis.

**Solution.** The slope of the parabola at $x = 1$ is $m = \frac{d}{dx} x^2 \big|_{x=1} = 2x \big|_{x=1} = 2$. This is also the slope of the tangent line to $y = x^2$ at $x = 1$. So the equation of the tangent line is $y – 1 = m(x – 1) = 2(x – 1)$ or $y = 2x – 1$. We are to determine the area of the region in the figure
Use horizontal slices. On each slice, $y$ is constant (with some fixed value between 0 and 1) and $x$ runs from $\sqrt{y}$ to $(y + 1)/2$. So, the area is

$$
\int_0^1 \left[(y + 1)/2 - \sqrt{y}\right] dy = \left[\frac{1}{2} \left[ y^2 + y \right] - \frac{y^{3/2}}{3/2} \right]_0^1 = \frac{1}{2} \left[ \frac{1}{2} + 1 \right] - 1^{3/2} \cdot \frac{3}{2} = \frac{3}{4} - \frac{3}{2} = \frac{1}{12}
$$

6) Find the number $b$ such that the line $y = b$ divides the region bounded by the curves $y = x^2$ and $y = 4$ into two regions with equal area.

**Solution.** The area of the part of region with $0 \leq y \leq b$ is

$$A_1 = \int_0^b \sqrt{y} \, dy = \left[ \frac{y^{3/2}}{3/2} \right]_0^b = \frac{2}{3} b^{3/2}$$

The area of the part of region with $b \leq y \leq 4$ is

$$A_2 = \int_b^4 \sqrt{y} \, dy = \left[ \frac{y^{3/2}}{3/2} \right]_b^4 = \frac{2}{3} 4^{3/2} - \frac{2}{3} b^{3/2}$$

The two areas are equal when

$$\frac{2}{3} b^{3/2} = \frac{2}{3} 4^{3/2} - \frac{2}{3} b^{3/2} \Rightarrow 2 \frac{2}{3} b^{3/2} = \frac{2}{3} 4^{3/2} \Rightarrow b^{3/2} = \frac{1}{2} 4^{3/2} \Rightarrow b = \frac{4}{2^{3/2}} = \frac{2^{2}}{2^{3/2}} = \frac{2^4}{3}$$

7) Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the specified curves about the given line.

a) $y = \ln x$, $y = 1$, $x = 1$, about the $x$–axis.

b) $y = \cos x$, $y = 0$, $x = 0$, $x = \frac{\pi}{2}$, about the line $y = 1$.

**Solution.** a) When the vertical strip shown in the figure is rotated about the $x$–axis it forms a washer with inner radius $\ln x$ and outer radius 1. The area of this washer is $\pi 1^2 - \pi (\ln x)^2$. As $x$ runs from 1 to $\pi$ (which is where $\ln x = 1$) the volume of the solid is $\pi \int_1^\pi [1 - (\ln x)^2] \, dx$.

b) When the vertical strip shown in the figure below is rotated about the line $y = 1$ it forms a washer with inner radius $1 - \cos x$ and outer radius 1. The area of this washer is

$$\pi 1^2 - \pi (1 - \cos x)^2$$
As $x$ runs from 0 to $\frac{\pi}{2}$, the volume of the solid is $\pi \int_{0}^{\pi/2} [1^2 - (1 - \cos x)^2] \, dx$.

8) Each of the following integrals represents the volume of a solid. Describe the solid.

a) $\pi \int_{1}^{2} y^6 \, dy$

b) $\pi \int_{0}^{4} [16 - (x - 2)^4] \, dx$

**Solution.**

a) When the horizontal strip shown in the figure on the left below is rotated about the $y$–axis it forms a disk with radius $f(y)$ and hence of area $\pi f(y)^2$. The volume of the region obtained when one rotates the curve $x = f(y)$, $a \leq y \leq b$ about the $y$–axis is $\pi \int_{a}^{b} f(y)^2 \, dy$. This matches the given integral when $a = 1$, $b = 2$ and $f(y) = y^3$. So the integral represents the volume of the region obtained when one rotates the curve $x = y^3$, $1 \leq y \leq 2$ about the $y$–axis.

b) The integrand $\pi [16 - (x - 2)^4] = \pi 4^2 - \pi [(x - 2)^2]^2$ is the area of a washer of outer radius 4 and inner radius $(x - 2)^2$. Suppose in general that the region $g(x) \leq y \leq f(x)$ is rotated about the $x$–axis. When the vertical strip shown in the figure on the right above is rotated about the $x$–axis it forms a washer with outer radius $f(x)$ and inner radius $g(x)$ and hence of area $\pi [f(x)^2 - g(x)^2]$. The volume of the region obtained when one rotates $g(x) \leq y \leq f(x)$, $a \leq x \leq b$ about the $x$–axis is $\pi \int_{a}^{b} [f(x)^2 - g(x)^2] \, dx$. This matches the given integral when $a = 0$, $b = 4$, $f(x) = 4$ and $g(x) = (x - 2)^2$. So the integral represents the volume of the region obtained when one rotates $(x - 2)^2 \leq y \leq 4$, $0 \leq x \leq 4$ about the $x$–axis.
9) Find the volume of a solid torus (donut) obtained by rotating the circle \((x-R)^2 + y^2 = r^2\) about the \(y\)-axis.

Solution.

The torus may be formed by rotating the circle in the figure on the left above about the \(y\)-axis. Call this circle the “generating circle”. It is centred on \((R,0)\) and has radius \(r\) and so has equation \((x-R)^2 + y^2 = r^2\). Slice the donut horizontally. The figure on the right shows a top view of a typical slice. The points on one slice all have the same value of \(y\) (with some \(-1 \leq y \leq 1\)). The slice looks like a washer. Our main problem is to determine the inner and outer radii of the washer. The portion of the washer that lies in the generating circle is marked in both figures as a horizontal line. The end points of the line are denoted \(P_1\) and \(P_2\). Both \(P_1\) and \(P_2\) lie on the washer and hence have \(y\)-coordinate \(y\). They also both lie on the generating circle. We may determine their \(x\)-coordinates by solving \((x-R)^2 + y^2 = r^2\) for \(x\). The \(x\)-coordinate of \(P_1\) is \(x = R - \sqrt{r^2 - y^2}\) that of \(P_2\) is \(x = R + \sqrt{r^2 - y^2}\). So the washer has inner radius \(R - \sqrt{r^2 - y^2}\) and outer radius \(R + \sqrt{r^2 - y^2}\). The area of the washer is

\[
\pi \left( R + \sqrt{r^2 - y^2} \right)^2 - \pi \left( R - \sqrt{r^2 - y^2} \right)^2 = \pi \left[ 4R \sqrt{r^2 - y^2} \right]
\]

The volume of the donut is the integral of the cross-sectional area of the slice from \(y = -r\) to \(y = r\). That is,

\[
4\pi R \int_{-r}^{r} \sqrt{r^2 - y^2} \, dy = 4\pi R \left( \text{area of right half of a circle of radius } r \right)
\]

\[
= 4\pi R \left( \frac{1}{2} \pi r^2 \right) = 2\pi^2 R r^2
\]

10) Evaluate \(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2 \sin x}{1 + x^2} \, dx\).

Solution. The critical observations here are that the integrand is odd and the domain of integration is symmetric about \(x = 0\). Recall that \(f(x)\) is odd if \(f(-x) = -f(x)\). I claim that, for any odd function \(f(x)\) and any positive number \(a\), \(\int_{-a}^{a} f(x) \, dx = 0\). Intuitively, here is the reason. The part of the graph \(y = f(x)\) from \(x = 0\) to \(x = a\) and part of the graph \(y = f(x)\) from \(x = 0\) to \(x = -a\) are just upside down images of
each other. So the magnitude of the area between the $x$–axis and the part of the graph $y = f(x)$ from $x = -a$ to $x = 0$ is the same as the magnitude of the area between the $x$–axis and the part of the graph $y = f(x)$ from $x = 0$ to $x = a$. The signed areas are negatives of each other. That is $\int_{-a}^{0} f(x) \, dx = -\int_{0}^{a} f(x) \, dx$, so that $\int_{-a}^{a} f(x) \, dx + \int_{0}^{a} f(x) \, dx = 0$. To see algebraically that $\int_{0}^{a} f(x) \, dx = -\int_{a}^{0} f(x) \, dx$, just make the substitution $x = -t, \, dx = -dt$:

$$\int_{0}^{a} f(x) \, dx = \int_{a}^{0} f(-t) (-dt) = -\int_{a}^{0} f(-t) \, dt = -\int_{0}^{a} f(t) \, dt$$

The oddness of $f$ was used in the last step. Choosing $f(x) = \frac{x^2 \sin x}{1+x^6}$ and $a = \frac{\pi}{2}$ gives $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2 \sin x}{1+x^6} \, dx = 0$.

11) Evaluate

a) $\int \frac{dx}{e^x + e^{-x}}$

b) $\int_{0}^{\pi/2} \sqrt{1 + \cos x} \, dx$

c) $\int_{0}^{\pi/2} \sqrt{1 - \sin x} \, dx$

Solution. a) Make the substitution $y = e^x, \, dy = e^x \, dx = y \, dx$.

$$\int \frac{dx}{e^x + e^{-x}} = \int \frac{1}{1/y + 1/y} \, dy = \int \frac{1}{1+y^2} \, dy = \tan^{-1} y + C = \tan^{-1} e^x + C$$

b) Using the trig identity $\cos 2\theta = 2 \cos^2 \theta - 1$ with $2\theta = x$

$$\int_{0}^{\pi/2} \sqrt{1 + \cos x} \, dx = \int_{0}^{\pi/2} \sqrt{2 \cos^2 \frac{x}{2}} \, dx$$

Since $\cos \frac{x}{2} \geq 0$ for all $0 \leq x \leq \frac{\pi}{2}$

$$\int_{0}^{\pi/2} \sqrt{2 \cos^2 \frac{x}{2}} \, dx = \int_{0}^{\pi/2} \sqrt{2} \cos \frac{x}{2} \, dx = \sqrt{2} \times 2 \sin \frac{\pi}{4} \bigg|_{0}^{\pi/2} = \sqrt{2} \times 2 \sin \frac{\pi}{4} = 2$$

b) (Second solution.)

$$\int_{0}^{\pi/2} \sqrt{1 + \cos x} \, dx = \int_{0}^{\pi/2} \sqrt{\frac{1+\cos x}{1-\cos x}} \, dx = \int_{0}^{\pi/2} \sqrt{\frac{1-\cos^2 x}{1-\cos x}} \, dx = \int_{0}^{\pi/2} \frac{\sin x}{\sqrt{1-\cos x}} \, dx$$

Now substitute $y = 1 - \cos x, \, dy = \sin x \, dx$

$$\int_{0}^{\pi/2} \frac{\sin x}{\sqrt{1-\cos x}} \, dx = \int_{0}^{1} \frac{dy}{\sqrt{y}} = 2\sqrt{y} \bigg|_{0}^{1} = 2$$
c) Make the substitution \( y = \frac{x}{2} - x, \ dy = -dx. \)

\[
\int_{0}^{\pi/2} \sqrt{1 - \sin x} \ dx = \int_{0}^{0} \sqrt{1 - \cos y} \ (-dy) = \int_{0}^{\pi/2} \sqrt{1 - \cos y} \ dy
\]

By the trig identity \( \cos 2\theta = 1 - 2\sin^2 \theta \) with \( \theta = \frac{y}{2} \),

\[
\int_{0}^{\pi/2} \sqrt{1 - \cos y} \ dy = \int_{0}^{\pi/2} \sqrt{2\sin^2 \frac{y}{2}} \ dy = \sqrt{2} \int_{0}^{\pi/2} \sin \frac{y}{2} \ dy = -2\sqrt{2} \cos \frac{y}{2} \bigg|_{0}^{\pi/2} = 2(\sqrt{2} - 1)
\]

12) Find the volume of the solid obtained by rotating the region bounded by \( y = 4x - x^2 \) and \( y = 8x - 2x^2 \) about \( x = -2 \).

**Solution.** The region bounded by \( y = 4x - x^2 \) and \( y = 8x - 2x^2 \) is sketched on the right. Note that the two parabolas meet at \((0,0)\) and \((4,0)\). Consider the thin slice in the figure on the right. It runs vertically from \((x, 4x - x^2)\) to \((x, 8x - 2x^2)\) and has width \( dx \). When this slice is rotated about \( x = -2 \), it sweeps out a cylindrical shell. A radius for the shell is shown in the figure on the right. It is the horizontal line half way up the thin slice. The \( x \)-coordinate of the right hand end of the radius is \( x \) and the \( x \)-coordinate of the left hand end is \(-2\). So the radius has length \( x - (-2) = x + 2 \). The height of the shell is the difference between the \( y \)-coordinates at the top and bottom of the thin slice. So the height of the shell is \( 8x - 2x^2 - (4x - x^2) = 4x - x^2 \). The thickness of the shell is \( dx \). So its volume is \( 2\pi(x+2)(4x - x^2)dx \). The total volume of the solid is

\[
\int_{0}^{4} 2\pi(x+2)(4x - x^2)dx = 2\pi \int_{0}^{4} (8x + 2x^2 - x^3) \ dx = 2\pi \left[4x^2 + \frac{2}{3}x^3 - \frac{1}{4}x^4\right]_{0}^{4} = 2\pi \left[64 + \frac{128}{3} - 64\right] = \frac{256}{3}\pi
\]

13) Find the volume of the solid obtained by rotating the region bounded by \( x = 4 - y^2 \) and \( x = 8 - 2y^2 \) about \( y = 5 \).

**Solution.** The region bounded by \( x = 4 - y^2 \) and \( x = 8 - 2y^2 \) is sketched below. Note that the two parabolas meet when \( 4 - y^2 = 8 - 2y^2 \) or \( y^2 = 4 \) or \( y = \pm2 \). The corresponding \( x = 4 - (\pm2)^2 = 0 \). So, the two parabolas meet at \((0, \pm2)\). Consider the thin slice in the figure on the right. It runs horizontally from \((4 - y^2, y)\) to \((8 - 2y^2, y)\) and has width \( dy \). When this slice is rotated about \( y = 5 \), it sweeps out a cylindrical shell, as illustrated in the figure on the left. A radius for the shell is shown in the figure on the right. It is the
vertical line half way along the thin slice. The $y$–coordinate of the top end of the radius is 5 and the $y$–coordinate of the bottom end is $y$. So the radius has length $5 - y$. The height of the shell is the difference between the $x$–coordinates at the right and left hand ends of the thin slice. So the height of the shell is $(8 - 2y^2) - (4 - y^2) = 4 - y^2$. The thickness of the shell is $dy$ and its volume is $2\pi (5 - y)(4 - y^2)dy$. The total volume of the solid is

$$
\int_{-2}^{2} 2\pi (5 - y)(4 - y^2)dy = 2\pi \int_{-2}^{2} (20 - 4y - 5y^2 + y^3) dy
$$

For any odd power $y^n$ of $y$, and any $a$, the integral $\int_{a}^{\alpha} y^n dy = 0$. This is because the area with $-a \leq y \leq 0$ has the same magnitude but opposite sign as the area with $0 \leq y \leq a$. See the figure on the left below. Thus the integrals $\int_{-2}^{2} y dy = \int_{-2}^{2} y^3 dy = 0$.

For any even power $y^n$ of $y$, and any $a$, the integral $\int_{a}^{\alpha} y^n dy = 2 \int_{0}^{a} y^n dy$. This is because the area with $-a \leq y \leq 0$ has the same magnitude and same sign as the area with $0 \leq y \leq a$. See the figure on the right above.

The volume of the solid is

$$
2\pi \int_{-2}^{2} (20 - 4y - 5y^2 + y^3) dy = 2\pi \int_{-2}^{2} (20 - 5y^2) dy = 4\pi \int_{0}^{2} (20 - 5y^2) dy
$$

$$
= 20\pi \int_{0}^{2} (4 - y^2) dy = 20\pi \left[4y - \frac{1}{3}y^3\right]_{0}^{2} = 20\pi \left[8 - \frac{8}{3}\right] = \frac{320}{3}\pi
$$

14) The region below the curve $y = \frac{1}{\sqrt{1-x^2}}$, above the $x$–axis and between the $y$–axis and the line $x = \frac{1}{2}$ is rotated about the line $x = 2$.
   a) Express the volume of the solid of revolution so obtained as a definite integral.
b) Evaluate the integral of part a.

Solution. a) When a point \((x, y)\), between the \(y\)-axis and the line \(x = \frac{1}{2}\), is rotated about the line \(x = 2\), it sweeps out a circle of radius \(2 - x\) and hence of circumference \(2\pi(2 - x)\). So, by the method of cylindrical shells,

\[
Vol = \int_0^{1/2} 2\pi (2 - x) \frac{1}{\sqrt{1-x^2}} \, dx
\]

b) \[
Vol = \int_0^{1/2} 2\pi (2 - x) \frac{1}{\sqrt{1-x^2}} \, dx = 4\pi \int_0^{1/2} \frac{1}{\sqrt{1-x^2}} \, dx - \pi \int_0^{1/2} \frac{2x}{\sqrt{1-x^2}} \, dx
\]

For the second integral, sub in \(t = 1 - x^2\), \(dt = -2xdx\).

\[
Vol = 4\pi \arcsin x \bigg|_0^{1/2} + \pi \int_1^{3/4} \frac{1}{\sqrt{t}} \, dt = 4\pi \left[ \frac{\sqrt{t}}{1/2} \right]_1^{3/4} = \frac{2\pi^2}{3} + \sqrt{3}\pi - 2\pi
\]

15) Let \(R\) be the plane region bounded by \(x = 0\), \(x = 1\), \(y = 0\) and \(y = c\sqrt{1 + x^2}\), where \(c\) is a positive constant.

(a) Find the volume \(V_1\) of the solid obtained by revolving \(R\) about the \(x\)-axis.

(b) Find the volume \(V_2\) of the solid obtained by revolving \(R\) about the \(y\)-axis.

(c) If \(V_1 = V_2\), what is the value of \(c\)?

Solution. a) Let \(V_1\) be the solid obtained by revolving \(R\) about the \(x\)-axis. The portion of \(V_1\) with \(x\)-coordinate between \(x\) and \(x + dx\) is a disk of radius \(c\sqrt{1 + x^2}\) and thickness \(dx\). The volume of this disk is \(\pi (c\sqrt{1 + x^2})^2 dx = \pi c^2(1 + x^2) dx\). The total volume of \(V_1\) is

\[
V_1 = \int_0^1 \pi c^2(1 + x^2) \, dx = \pi c^2 \left[ x + \frac{x^3}{3} \right]_0^1 = \frac{4\pi c^2}{3}
\]

b) Let \(V_2\) be the solid obtained by revolving \(R\) about the \(y\)-axis. Then \(V_2\) can be thought of as a union of cylindrical shells. The shell whose intersection with the \(x\)-axis is the interval from \(x\) to \(x + dx\) has radius \(x\), height \(c\sqrt{1 + x^2}\) and thickness \(dx\). The volume of this shell is \(2\pi x(c\sqrt{1 + x^2}) \, dx = 2\pi cx\sqrt{1 + x^2} \, dx\). The total volume of \(V_2\) is (making the substitution \(y = 1 + x^2\), \(dy = 2x \, dx\))

\[
V_2 = \int_0^1 2\pi cx\sqrt{1 + x^2} \, dx = \int_1^2 \pi c\sqrt{y} \, dy = \pi c \frac{y^{3/2}}{3/2} \bigg|_1^2 = \frac{2\pi c[2^{3/2} - 1]}{3}
\]

c) If \(V_1 = V_2\),

\[
\frac{4\pi c^2}{3} = \frac{2\pi c[2^{3/2} - 1]}{3} \implies c = 0 \text{ or } c = \frac{1}{2}[2^{3/2} - 1]
\]

16) A 45° notch is cut to the centre of a cylindrical log having radius 20 cm. One plane face of the notch is perpendicular to the axis of the log. What volume of wood was removed?
**Solution.** Slice the notch into rectangles as in the figure on the right below.

![Diagram of a notch sliced into rectangles](image)

Suppose that the base of the notch is in the $xy$–plane. Then the circular part of the boundary of the base of the notch has equation $x^2 + y^2 = 20^2$. If our coordinate system is such that $x$ is constant on each slice, then the slice has width $2y = 2\sqrt{20^2 - x^2}$ and height $x$ (since the upper face of the notch is at $45^\circ$ to the base). So the slice has cross–sectional area $2x\sqrt{20^2 - x^2}$ and the volume is

$$V = \int_0^{20} 2x\sqrt{20^2 - x^2} \, dx$$

Make the change of variables $t = 20^2 - x^2$, $dt = -2x \, dx$.

$$V = \int_0^{20^2} \sqrt{t} (-dt) = -\frac{t^{3/2}}{3/2} \bigg|_0^{20^2} = \frac{2}{3} 20^3 = \frac{16,000}{3}$$

**Solution 2** Suppose that the base of the notch is in the $xy$–plane with the skinny edge along the $y$–axis. Slice the notch into triangles parallel to the $x$–axis as in the figure below.

![Diagram of a notch sliced into triangles](image)

Then the circular part of the boundary of the base of the notch has equation $x^2 + y^2 = 20^2$. Our coordinate system is such that $y$ is constant on each slice, so that the slice has both base and height $x = \sqrt{20^2 - y^2}$ (since the upper face of the notch is at $45^\circ$ to the base). So the slice has cross–sectional area $\frac{1}{2} (\sqrt{20^2 - y^2})^2$ and the volume is

$$V = \frac{1}{2} \int_{-20}^{20} (20^2 - y^2) \, dy = \int_0^{20} (20^2 - y^2) \, dy = \left[ 20^2y - \frac{y^3}{3} \right]_0^{20} = \frac{2}{3} 20^3 = \frac{16,000}{3}$$

17) Evaluate the integrals

a) $\int x \tan^{-1} x \, dx$

b) $\int \sin(\ln x) \, dx$

c) $\int_1^4 e^{\sqrt{x}} \, dx$
Solution. a) Integrate by parts with \( u = \tan^{-1} x \) and \( dv = x \, dx \). Then \( du = \frac{1}{1+x^2} \, dx \) and \( v = \frac{1}{2} x^2 \), so

\[
\int x \tan^{-1} x \, dx = \frac{1}{2} x^2 \tan^{-1} x - \frac{1}{2} \int \frac{x^2}{1+x^2} \, dx
\]

The integral on the right hand side is

\[
\int \frac{x^2}{1+x^2} \, dx = \int \frac{1+x^2-1}{1+x^2} \, dx = \int dx - \int \frac{1}{1+x^2} \, dx = x - \tan^{-1} x + C
\]

All together

\[
\int x \tan^{-1} x \, dx = \frac{1}{2} x^2 \tan^{-1} x + \frac{1}{2} \tan^{-1} x - \frac{1}{2} x + C'
\]

with \( C' = \frac{1}{2} C \).

b) First, substitute \( y = \ln x \), \( dy = \frac{1}{x} \, dx \), in order to simplify the argument of \( \sin \). Because \( dx = e^y \, dy \),

\[
\int \sin(\ln x) \, dx = \int e^y \sin y \, dy
\]

Integrate by parts using \( u = e^y \) and \( dv = \sin y \, dy \) so that \( du = e^y \, dy \) and \( v = -\cos y \).

\[
\int e^y \sin y \, dy = -e^y \cos y - \int (-\cos y)e^y \, dy = -e^y \cos y + \int e^y \cos y \, dy
\]

Integrate by parts a second time using \( u = e^y \) and \( dv = \cos y \, dy \) so that \( du = e^y \, dy \) and \( v = \sin y \).

\[
\int e^y \sin y \, dy = -e^y \cos y + \int e^y \cos y \, dy = -e^y \cos y + e^y \sin y - \int (\sin y)e^y \, dy
\]

Moving the \( - \int (\sin y)e^y \, dy \) from the right hand side to the left hand side,

\[
2 \int e^y \sin y \, dy = -e^y \cos y + e^y \sin y \quad \implies \quad \int e^y \sin y \, dy = \frac{1}{2} \left[ -e^y \cos y + e^y \sin y \right]
\]

All together

\[
\int \sin(\ln x) \, dx = \int e^y \sin y \, dy = \frac{1}{2} \left[ e^y \sin y - e^y \cos y \right] + C
\]

\[
= \frac{1}{2} \left[ x \sin(\ln x) - x \cos(\ln x) \right] + C
\]

c) First, substitute \( y = \sqrt{x} \), \( dy = \frac{1}{2\sqrt{x}} \, dx \), in order to simplify the argument of the exponential. Because \( dx = 2\sqrt{x} \, dy = 2y \, dy \) and \( y(1) = 1 \), \( y(4) = 2 \),

\[
\int_1^4 e^{\sqrt{x}} \, dx = \int_1^2 e^y \, 2y \, dy = 2 \int_1^2 ye^y \, dy
\]
Integrate by parts using \( u = y \) and \( dv = e^y \, dy \) so that \( du = dy \) and \( v = e^y \).

\[
\int_1^4 e^{\sqrt{x}} \, dx = 2 \int ye^y \, dy = 2 \left[ ye^y - \int_1^2 e^y \, dy \right] = 2 \left[ ye^y - e^y \right]_1^2 \\
= 2 \left[ (2e^2 - e^2) - (1e^1 - e^1) \right] = 2e^2
\]

18) Evaluate \( \int \frac{dx}{1 - \sin x} \).

**Solution.**

\[
\int \frac{dx}{1 - \sin x} = \int \frac{1 + \sin x}{1 - \sin^2 x} \, dx = \int \frac{1 + \sin x}{\cos^2 x} \, dx = \int \sec^2 x \, dx + \int \frac{\sin x}{\cos^2 x} \, dx \\
= \tan x - \int \frac{d(\cos x)}{\cos^2 x} = \tan x + \frac{1}{\cos x} + C
\]

**Solution 2** Using the trig identities \( \sin \left( \frac{\pi}{2} - y \right) = \cos y \) and \( \cos(2y) = 1 - 2\sin^2 y \),

\[
\int \frac{dx}{1 - \sin x} = \int \frac{1}{1 - \cos(\frac{\pi}{2} - x)} \, dx = \int \frac{1}{1 - [1 - 2\sin^2(\frac{\pi}{4} - \frac{x}{2})]} \, dx = \frac{1}{2} \int \frac{1}{\sin^2(\frac{\pi}{4} - \frac{x}{2})} \, dx \\
= \cot(\frac{\pi}{4} - \frac{x}{2}) + C
\]

The answers in the two solutions are really the same. Using the trig identities \( \sin(a - b) = \sin a \cos b - \cos a \sin b \) and \( \cos(a - b) = \cos a \cos b + \sin a \sin b \),

\[
\cot(\frac{\pi}{4} - \frac{x}{2}) = \frac{\cos(\frac{\pi}{4} - \frac{x}{2})}{\sin(\frac{\pi}{4} - \frac{x}{2})} = \frac{\cos \frac{\pi}{4} \cos \frac{x}{2} + \sin \frac{\pi}{4} \sin \frac{x}{2}}{\sin \frac{\pi}{4} \cos \frac{x}{2} - \cos \frac{\pi}{4} \sin \frac{x}{2}}
\]

Since \( \sin \frac{\pi}{4} = \cos \frac{\pi}{4} \)

\[
\cot(\frac{\pi}{4} - \frac{x}{2}) = \frac{\cos \frac{\pi}{4} + \sin \frac{x}{2}}{\cos \frac{\pi}{4} - \sin \frac{x}{2}} = \frac{[\cos \frac{\pi}{4} + \sin \frac{x}{2}]^2}{\cos \frac{\pi}{4} - \sin \frac{x}{2}} = \frac{1 + 2 \sin \frac{x}{2} \cos \frac{\pi}{4}}{\cos \frac{x}{2}} = \frac{1 + \sin x}{\cos x} = \sec x + \tan x
\]

19) Household electricity is supplied in the form of alternating current that varies from \(-155\) V to \(155\) V with a frequency of 60 cycles per second. The voltage is given by the equation

\[
V(t) = 155 \sin(120\pi t)
\]

where \( t \) is the time in seconds. Voltmeters read the RMS (root-mean-square) voltage, which is the square root of the average value of \([V(t)]^2\) over one cycle. Calculate the RMS voltage of household current.
Solution. The voltage goes through 60 cycles in one second. So one cycle takes \( \frac{1}{60} \) sec and

\[
\text{RMS voltage} = \sqrt{ \frac{1}{1/60} \int_0^{1/60} V(t)^2 \, dt } = \sqrt{ 60 \int_0^{1/60} 155^2 \sin^2(120\pi t) \, dt }
\]

\[
= \sqrt{ 60(155)^2 \int_0^{1/60} \frac{1 - \cos(2\times120\pi t)}{2} \, dt }
\]

\[
= \sqrt{ \frac{1}{2} 60(155)^2 \left[ t - \frac{\sin(2\times120\pi t)}{240\pi} \right]_0^{1/60} } = \sqrt{ \frac{1}{2} 60(155)^2 \frac{1}{60} }
\]

\[
= \frac{155}{\sqrt{2}} \approx 109.6 \text{ V}
\]

20) If a freely falling body starts from rest, then its displacement is given by \( s = \frac{1}{2}gt^2 \). Let the velocity after a time \( T \) be \( v_T \). Show that if we compute the average of the velocities with respect to \( t \) we get \( v_{ave} = \frac{1}{2}v_T \), but if we compute the average of the velocities with respect to \( s \) we get \( v_{ave} = \frac{2}{3}v_T \).

Solution. The speed at time \( t \) is \( ds/dt = gt \). In particular \( v_T = gT \). The average velocity over the time interval \( 0 \leq t \leq T \) is

\[
\frac{1}{T} \int_0^T gt \, dt = \frac{1}{T} \left[ \frac{1}{2}gt^2 \right]_0^T = \frac{1}{T} \frac{1}{2}gT^2 = \frac{1}{2}gT = \frac{1}{2}v_T
\]

The body reaches displacement \( s \) at the time \( t \) that obeys \( s = \frac{1}{2}gt^2 \). That is, at \( t = \sqrt{2s/g} \). At displacement \( s \), the velocity of the body is \( g\sqrt{2s/g} = \sqrt{2sg} \). The average velocity over the displacement interval \( 0 \leq s \leq \frac{1}{2}gT^2 \) (the final displacement is the value of \( s \) at time \( T \)) is

\[
\frac{1}{\sqrt{T^2/2}} \int_0^{T^2/2} \sqrt{2sg} \, ds = \frac{1}{\sqrt{T^2/2}} \left[ \sqrt{2gs^{3/2}} \right]_0^{T^2/2} = \frac{1}{\sqrt{T^2/2}} \sqrt{2g\left(\frac{gT^2/2}{3/2}\right)^{3/2}} = \frac{1}{\sqrt{T^2/2}} \frac{gT^3}{3} = \frac{2}{3}gT = \frac{2}{3}v_T
\]

21) Show that if \( f(a) = f(b) = 0 \) and \( f \) is twice differentiable, then

\[
\int_a^b (x-a)(b-x) f''(x) \, dx = -2 \int_a^b f(x) \, dx
\]

Solution. Integrating by parts with \( u = (x-a)(b-x), \, du = [(b-x)-(x-a)] \, dx, \)
\( dv = f''(x) \, dx \) and \( v = f'(x) \)

\[
\int_a^b (x-a)(b-x) f''(x) \, dx = (x-a)(b-x)f'(x) \bigg|_a^b - \int_a^b (a+b-2x)f'(x) \, dx
\]

\[
= -\int_a^b (a+b-2x)f'(x) \, dx
\]
Integrating by parts with \( u = a + b - 2x, \) \( du = -2 \, dx, \) \( dv = f'(x) \, dx \) and \( v = f(x) \)

\[
\int_a^b (x-a)(b-x)f''(x) \, dx = - \int_a^b (a+b-2x)f'(x) \, dx \\
= -(a+b-2x)f(x) \bigg|_a^b - 2 \int_a^b f(x) \, dx \\
= -2 \int_a^b f(x) \, dx
\]

22) Find the volume enclosed by the ellipsoid

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1
\]

**Solution.** Suppose that we compute the volume by slicing it into horizontal disks.

The height \( z \) of a disk varies from \( z = -c \) to \( z = c \). The disk at height \( z \) has equation \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{z^2}{c^2} \). This is an ellipse with semi-axes \( a\sqrt{1 - \frac{z^2}{c^2}} \) and \( b\sqrt{1 - \frac{z^2}{c^2}} \). Hence the cross-sectional area of the disk is

\[
\pi a \sqrt{1 - \frac{z^2}{c^2}} b \sqrt{1 - \frac{z^2}{c^2}} = \pi ab \left(1 - \frac{z^2}{c^2}\right).
\]

The volume of the ellipsoid is

\[
V = \int_{-c}^{c} \pi ab \left(1 - \frac{z^2}{c^2}\right) \, dz = 2\pi ab \left[ z - \frac{z^3}{3c^2} \right]_{-c}^{c} = \frac{4}{3} \pi abc
\]

23) Evaluate the following integrals.

a) \( \int_0^2 x^3 \sqrt{4 - x^2} \, dx \) \quad b) \( \int \frac{dx}{\sqrt{x^2 + 4x + 8}} \) \quad c) \( \int \frac{dx}{(5-4x-x^2)^{3/2}} \)

**Solution.**

a) Make the substitution \( u = 4 - x^2 \). Then \( du = -2x \, dx \) so that

\[
x^3 \sqrt{4 - x^2} \, dx = x^2 \sqrt{4 - x^2} \, dx = (4 - u) \sqrt{u} \, du.
\]

When \( x = 0, \) \( u = 4 \) and when \( x = 2, \) \( u = 0 \). Hence

\[
\int_0^2 x^3 \sqrt{4 - x^2} \, dx = \int_4^0 (4-u) \sqrt{u} \, du = \int_4^0 (\sqrt{u} - 2\sqrt{u} + \frac{1}{2}u^{3/2}) \, du = \left[ \frac{4}{3} u^{3/2} + \frac{1}{5} u^{5/2} \right]_4^0
\]

\[
= \frac{4}{3} \, 8 - \frac{1}{5} \, 32 = (\frac{1}{3} - \frac{1}{5}) \, 32 = \frac{64}{15}
\]
b) \[ \int \frac{dx}{\sqrt{x^2+4x+8}} = \int \frac{dx}{\sqrt{x^2+4x+4+4}} = \int \frac{dx}{\sqrt{(x+2)^2+4}} \]

Sub in \( x+2 = 2\tan \theta \). Then \( dx = 2\sec^2 \theta \, d\theta \) and \( \sqrt{(x+2)^2+4} = \sqrt{4\tan^2 \theta + 4} = \sqrt{4\sec^2 \theta} = 2\sec \theta \). So \[
\int \frac{dx}{\sqrt{x^2+4x+8}} = \int \frac{2\sec^2 \theta \, d\theta}{2\sec \theta} = \int \sec \theta \, d\theta = \ln |\sec \theta + \tan \theta| + C
\]

Subbing in \( \theta \) in terms of \( x \)

\[
\int \frac{dx}{\sqrt{x^2+4x+8}} = \ln \left| \frac{\sqrt{x^2+4x+8} + \frac{x+2}{2}}{x+2} \right| + C
\]

c) \[ \int \frac{dx}{(5-4x-x^2)^{5/2}} = \int \frac{dx}{(9-4-4x-x^2)^{5/2}} = \int \frac{dx}{(9-(2+x)^2)^{5/2}} \]

Sub in \( x+2 = 3\sin \theta \). Then \( dx = 3\cos \theta \, d\theta \) and \( (9-(x+2)^2)^{5/2} = (9-9\sin^2 \theta)^{5/2} = (9\cos^2 \theta)^{5/2} = 3^5\cos^5 \theta \). So \[
\int \frac{dx}{(5-4x-x^2)^{5/2}} = \int \frac{3\cos \theta \, d\theta}{3^5\cos^6 \theta} = \frac{1}{81} \int \frac{d\theta}{\cos^4 \theta} = \frac{1}{81} \int \sec^2 \theta \sec^2 \theta \, d\theta
\]

\[
= \frac{1}{81} \int [1 + \tan^2 \theta] \sec^2 \theta \, d\theta
\]

Now sub in \( y = \tan \theta \). Then \( dy = \sec^2 \theta \, d\theta \) so \[
\int \frac{dx}{(5-4x-x^2)^{5/2}} = \frac{1}{81} \int [1 + y^2] \, dy = \frac{1}{81} \left[ y + \frac{y^3}{3} \right] + C
\]

Subbing back in \( y = \tan \theta \),

\[
\int \frac{dx}{(5-4x-x^2)^{5/2}} = \frac{1}{81} \left[ \tan \theta + \frac{\tan^3 \theta}{3} \right] + C
\]

Finally we have sub back in \( \theta \) in terms of \( x \), using \( x+2 = 3\sin \theta \). From the triangle of the right, \( \tan \theta = \frac{x+2}{\sqrt{9-(x+2)^2}} = \frac{x+2}{\sqrt{5-4x-x^2}} \) so \[
\int \frac{dx}{(5-4x-x^2)^{5/2}} = \frac{1}{81} \left[ \frac{x+2}{\sqrt{5-4x-x^2}} + \frac{(x+2)^3}{3(5-4x-x^2)^{3/2}} \right] + C
\]
24) Find the area of the crescent shaped region bounded by arcs of circles with radii \(r\) and \(R\), as in the figure to the right.

**Solution.** The large circle has equation \(x^2 + y^2 = R^2\). The small one is centred at \(x = 0, y = \sqrt{R^2 - r^2}\) and has radius \(r\), so its equation is \(x^2 + (y - \sqrt{R^2 - r^2})^2 = r^2\). The area is

\[
\int_{-r}^{r} \left[ \sqrt{R^2 - r^2} + \sqrt{r^2 - x^2} - \sqrt{R^2 - x^2} \right] \, dx
\]

\[
= \int_{-r}^{r} \sqrt{R^2 - r^2} \, dx + \int_{-r}^{r} \sqrt{r^2 - x^2} \, dx - \int_{-r}^{r} \sqrt{R^2 - x^2} \, dx
\]

The first integral represents the area of a rectangle of width \(2r\) and height \(\sqrt{R^2 - r^2}\). So the first integral equals \(2r\sqrt{R^2 - r^2}\). The second integral represents the area of the top half of a circle of radius \(r\). So the second integral equals \(\frac{1}{2} \pi r^2\). For the third integral substitute \(x = R \sin \theta\). Then \(dx = R \cos \theta \, d\theta\). When \(x = \pm r\), \(\theta = \pm \sin^{-1} \frac{r}{R}\). So

\[
\int_{-r}^{r} \sqrt{R^2 - x^2} \, dx = \int_{-\sin^{-1} \frac{r}{R}}^{\sin^{-1} \frac{r}{R}} R^2 \cos^2 \theta \, d\theta = 2R^2 \int_{0}^{\sin^{-1} \frac{r}{R}} \frac{1 + \cos 2\theta}{2} \, d\theta
\]

\[
= R^2 \left[ \theta + \frac{\sin 2\theta}{2} \right]_{0}^{\sin^{-1} \frac{r}{R}} = R^2 \left[ \theta + \sin \theta \cos \theta \right]_{0}^{\sin^{-1} \frac{r}{R}}
\]

When \(\theta = \sin^{-1} \frac{r}{R}\), \(\sin \theta = \frac{r}{R}\) and \(\cos \theta = \sqrt{1 - \frac{r^2}{R^2}}\) so

\[
\int_{-r}^{r} \sqrt{R^2 - x^2} \, dx = R^2 \left[ \sin^{-1} \frac{r}{R} + \frac{r}{R} \sqrt{1 - \frac{r^2}{R^2}} \right]
\]

and the area is

\[
2r\sqrt{R^2 - r^2} + \frac{1}{2} \pi r^2 - R^2 \sin^{-1} \frac{r}{R} - R^2 \frac{r}{R} \sqrt{1 - \frac{r^2}{R^2}} = r\sqrt{R^2 - r^2} + \frac{1}{2} \pi r^2 - R^2 \sin^{-1} \frac{r}{R}
\]

25) Let \(R\) be the region in the \(xy\)-plane bounded by \(y = x \sin x\), \(y = 0\), \(x = 0\) and \(x = \pi\).

a) Find the volume of the solid obtained by rotating \(R\) about the \(y\)-axis.

b) Find the volume of the solid obtained by rotating \(R\) about the \(x\)-axis.

**Solution.** a) By cylindrical shells,

\[
\text{Vol} = \int_{0}^{\pi} 2\pi x (x \sin x) \, dx = 2\pi \int_{0}^{\pi} x^2 \sin x \, dx
\]

17
By integration by parts with $u = x^2$, $dv = \sin x \, dx$, $du = 2x \, dx$, $v = -\cos x$

\[
\text{Vol} = 2\pi x^2(-\cos x) \bigg|_0^\pi + 4\pi \int_0^\pi x \cos x \, dx = 2\pi^3 + 4\pi \int_0^\pi x \cos x \, dx
\]

By integration by parts, a second time, with $u = x$, $dv = \cos x \, dx$, $du = dx$, $v = \sin x$

\[
\text{Vol} = 2\pi^3 + 4\pi x \sin x \bigg|_0^\pi - 4\pi \int_0^\pi \sin x \, dx = 2\pi^3 - 4\pi(-\cos x) \bigg|_0^\pi = 2\pi^3 - 8\pi
\]

b)

\[
\text{Vol} = \int_0^\pi \pi (x \sin x)^2 \, dx = \pi \int_0^\pi x^2 \frac{1-\cos 2x}{2} \, dx
\]

\[
= \frac{\pi}{2} \left[ \frac{x^3}{3} - \frac{1}{2} x^2 \sin 2x - \frac{1}{2} x \cos 2x + \frac{1}{4} \sin 2x \right]_0^\pi = \frac{\pi}{2} \left[ \frac{\pi^3}{3} - \frac{1}{2} \pi \right]
\]

\[
= \frac{\pi^2}{2} \left[ \frac{\pi^2}{3} - \frac{1}{2} \right]
\]

26) Let $R$ be the region bounded by the line $y = 2$ and $y = (x - 4)^2 - 2$. Calculate the position of the centroid of $R$.

**Solution.** The region is symmetric about $x = 4$, so $\bar{x} = 4$. The line and parabola intersect when $(x - 4)^2 - 2 = 2$ or $(x - 4)^2 = 4$ or $x - 4 = \pm 2$ or $x = 2, 6$. The area of $R$ is

\[
A = \int_2^6 \left[ 2 - [(x - 4)^2 - 2] \right] dx = \int_2^6 \left[ -x^2 + 8x - 12 \right] dx
\]

\[
= \left[ -\frac{x^3}{3} + 4x^2 - 12x \right]_2^6 = 0 - \left[ -\frac{32}{3} \right] = \frac{32}{3}
\]

We use vertical approximating rectangle to compute $\bar{y}$. A typical rectangle has width $dx$ and $y$ running from $(x - 4)^2 - 2$ to 2. Thus the average value of $y$ on the rectangle is

\[
\frac{1}{2} \left[ 2 + [(x - 4)^2 - 2] \right] = \frac{1}{2} (x - 4)^2
\]

and the area of the rectangle is $\left[ -x^2 + 8x - 12 \right] dx$.

\[
\bar{y} = \frac{1}{A} \int_2^6 \frac{1}{2} (x - 4)^2 \left[ -x^2 + 8x - 12 \right] dx = \frac{1}{A} \int_2^6 \frac{1}{2} (x - 4)^2 (6 - x)(x - 2) \, dx
\]

Sub in $x = t + 4$, $dx = dt$.

\[
\bar{y} = \frac{1}{A} \int_{-2}^2 \frac{1}{2} t^2 (2 - t)(t + 2) \, dt = \frac{3}{64} \int_{-2}^2 [4t^2 - t^4] \, dt = \frac{3}{32} \int_0^2 [4t^2 - t^4] \, dt
\]

\[
= \frac{3}{32} \left[ \frac{4t^3}{3} - \frac{t^5}{5} \right]_0^2 = \frac{3}{32} \left[ \frac{32}{3} - \frac{32}{5} \right] = \frac{2}{5}
\]
27) Find the $x$-coordinate of the centroid (centre of gravity) of the plane region $R$ that lies in the first quadrant $x \geq 0, y \geq 0$ and inside the ellipse $4x^2 + 9y^2 = 36$. (The area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\pi ab$ square units.)

**Solution.** In standard form $4x^2 + 9y^2 = 36$ is $\frac{x^2}{9} + \frac{y^2}{4} = 1$. The ellipse has $a = 3$ and $b = 2$. So, on $R$, $x$ runs from 0 to $a = 3$ and $R$ has area $A = \frac{1}{4}\pi \times 3 \times 2 = \frac{3}{2}\pi$. For each fixed $x$, between 0 and 3, $y$ runs from 0 to $y(x) = 2\sqrt{1 - \frac{x^2}{9}}$.

$$\bar{x} = \frac{1}{A} \int_0^3 x y(x) \, dx = \frac{1}{A} \int_0^3 x 2\sqrt{1 - \frac{x^2}{9}} \, dx = \frac{4}{3\pi} \int_0^3 x \sqrt{1 - \frac{x^2}{9}} \, dx$$

Sub in $t = 1 - \frac{x^2}{9}$, $dt = -\frac{2}{9}x \, dx$.

$$\bar{x} = -\frac{9}{2} \frac{4}{3\pi} \int_1^0 \sqrt{t} \, dt = -\frac{9}{2} \frac{4}{3\pi} \left[ \frac{t^{3/2}}{3/2} \right]_1^0 = -\frac{9}{2} \frac{4}{3\pi} \left[ -\frac{2}{3} \right] = \frac{4}{\pi}$$

28) a) Find the centroid of the quarter circular disk $x \geq 0, y \geq 0, x^2 + y^2 \leq a^2$.

b) Find the centroid of the region $R$ in the diagram.

**Solution.** a) By symmetry, $\bar{x} = \bar{y}$. The area of the quarter disk is $A = \frac{1}{4}\pi a^2$.

$$\bar{x} = \frac{1}{A} \int_0^a x \sqrt{a^2 - x^2} \, dx$$

To evaluate the integral, sub in $t = a^2 - x^2$, $dt = -2x \, dx$.

$$\int_0^a x \sqrt{a^2 - x^2} \, dx = \int_{a^2}^0 \sqrt{t} \, \frac{dt}{-2} = -\frac{1}{2} \left[ \frac{t^{3/2}}{3/2} \right]_{a^2}^0 = \frac{a^3}{3}$$

So

$$\bar{x} = \frac{4}{\pi a^2} \left[ \frac{a^3}{3} \right] = \frac{4a}{3\pi}$$
b) Again, by symmetry, \( \bar{x} = \bar{y} \). The area of the region is \( A = 2 \times 2 - \frac{1}{4} \pi = \frac{16-\pi}{4} \).

\[
\bar{x} = \frac{1}{A} \left[ \int_0^1 x[2 - \sqrt{1-x^2}] \, dx + \int_1^2 x[2] \, dx \right] \\
= \frac{4}{16-\pi} \left[ x^2 \bigg|_0^1 + x^2 \bigg|_1^2 - \int_0^1 x \sqrt{1-x^2} \, dx \right] \\
= \frac{4}{16-\pi} \left[ 4 - \frac{1}{3} \right] \text{ by (*) with } a = 1 \\
= \frac{44}{48-3\pi}
\]

**Solution 2** When the quarter disk is rotated about the \( y \)-axis a hemisphere is formed. The hemisphere has radius \( a \) and hence volume \( \frac{1}{2} \pi a^3 = \frac{2}{3} \pi a^3 \). The quarter disk has area \( \frac{1}{4} \pi a^2 = \frac{1}{4} \pi a^2 \). By Pappus’s Theorem \( \frac{2}{3} \pi a^3 = V = 2\pi \bar{x} A = \frac{1}{2} \pi^2 a^2 \bar{x} \), so that \( \bar{x} = \frac{4a}{3\pi} \). To get \( \bar{y} = \bar{x} \) either invoke symmetry or rotate about the \( x \)-axis.

When \( R \) is rotated about the \( y \)-axis a cylinder with a hemispherical hole is formed. The cylinder has radius 2 and height 2 and hence volume \( \pi \times 2^2 \times 2 \) and the hole has radius 1 and hence volume \( \frac{1}{4} \frac{4}{3} \pi \times 1^2 \). So the volume of the solid formed by rotating \( R \) is \( \frac{22}{3} \pi \).

\( R \) has area \( 2^2 - \frac{1}{3} \pi \). By Pappus’s Theorem \( \bar{x} = \frac{V}{2\pi A} = \frac{22\pi/3}{2\pi(16-\pi)/4} = \frac{44}{48-3\pi} \). Again, to get \( \bar{y} = \bar{x} \) either invoke symmetry or rotate about the \( x \)-axis.

29) Prove that the centroid of any triangle is located at the point of intersection of the medians.

**Solution.** Choose a coordinate system so that the vertices of the triangle are located at \((a,0), (0,b)\) and \((c,0)\). (In the figure, \(a\) is negative.)

![Triangle](image)

The line joining \((a,0)\) and \((0,b)\) has equation \( bx + ay = ab \). (Check that \((a,0)\) and \((0,b)\) both really are on this line.) The line joining \((c,0)\) and \((0,b)\) has equation \( bx + cy = bc \). (Check that \((c,0)\) and \((0,b)\) both really are on this line.) Hence for each fixed \( y \) between 0 and \( b \), \( x \) runs from \( a - \frac{c}{b} y \) to \( c - \frac{c}{b} y \). A thin strip at height \( y \) has length \( \ell(y) = \left[ (c - \frac{c}{b} y) - (a - \frac{c}{b} y) \right] = \frac{c-a}{b} (b-y) \). On this strip \( y \) has average value \( \frac{1}{2} [(a - \frac{c}{b} y) + (c - \frac{c}{b} y)] = \frac{a+c}{2b} (b-y) \). As the area of the triangle is \( A = \frac{1}{2}(c-a)b \)

\[
\bar{y} = \frac{1}{A} \int_0^b y \ell(y) \, dy = \frac{2}{(c-a)b} \int_0^b y \frac{c-a}{b} (b-y) \, dy = \frac{2}{b^2} \int_0^b (by - y^2) \, dy = \frac{2}{b^2} \left( \frac{b^2}{2} - \frac{b^3}{3} \right)
\]

\[
= \frac{2}{b^2} \frac{b^3}{6} = \frac{b}{3}
\]
\[
\bar{x} = \frac{1}{A} \int_0^b \frac{a+c}{2b} (b-y) \ell(y) \, dy = \frac{2}{(c-a)b} \int_0^b \frac{a+c}{2b} (b-y) \left( \frac{c-a}{b} (b-y) \right) \, dy = \frac{a+c}{b^3} \int_0^b (y-b)^2 \, dy
\]

The midpoint of the line joining \((0, b)\) and \((c, 0)\) is \(\frac{1}{2} (c, b)\). The point two thirds of the way from \((a, 0)\) to \(\frac{1}{2} (c, b)\) is \(\frac{2}{3} (a, 0) + \frac{1}{3} (\frac{a+c}{3}, \frac{b^3}{3}) = (\bar{x}, \bar{y})\) as desired.

30) A water storage tanker has the shape of a cylinder with diameter 10 ft. It is mounted so that the circular cross-sections are vertical. If the depth of the water is 7 ft, what percentage of the total capacity is being used?

**Solution.** Here is an end view of the tank. The shaded part of the circle is filled with water. The cross-sectional area of the the filled part of the tank is the area of a circle of radius 5, namely \(\pi 5^2\), minus the area of the unfilled cap. The area of the cap is

\[
\text{Area}\left(\text{\begin{array}{c} \theta \\ 5 \end{array}}\right) = \text{Area}\left(\text{\begin{array}{c} 2\theta \\ 5 \end{array}}\right) - \text{Area}\left(\text{\begin{array}{c} \theta \\ 5 \end{array}}\right)
\]

The first term is the fraction \(\frac{2\theta}{2\pi}\) of the area of a circle of radius 5. The triangle in the second has base \(2 \times 5 \cos \theta\) and height \(5 \sin \theta\). So the area of the cap is

\[
\frac{2\theta}{2\pi} \pi 5^2 - \frac{1}{2} (2 \times 5 \cos \theta) (5 \sin \theta) = 25\theta - 25 \sin \theta \cos \theta
\]

The tank is filled to a height of 7 ft, which is 2 ft above the centre of the tank. So \(\cos \theta = \frac{2}{5} = 0.4\) and \(\sin \theta = \sqrt{1 - \frac{4}{25}} \approx 0.917\) and \(\theta = \arccos \frac{2}{5} \approx 1.159\). The area of the cap is \(25[1.159 - 0.4 \times 0.917] \approx 19.82\). The area of the full circle is \(A = 25\pi\), so the fraction filled is

\[
\frac{A - 19.82}{A} = \frac{58.72}{78.54} = 0.74 = 74.8\%
\]

31) Evaluate the following integrals.

a) \(\int \frac{18 - 2x - 4x^2}{x^3 + 4x^2 + x - 6} \, dx\)  

b) \(\int \frac{x^3}{(x+1)^x} \, dx\)  

c) \(\int \frac{2x^3 - x}{x^4 - x^2 + 1} \, dx\)  

d) \(\int_0^1 \sqrt{x^2 + 1} \, dx\)
**Solution.** a) First, we have to factor the denominator $P(x) = x^3 + 4x^2 + x - 6$. This is a polynomial with integer coefficients. So any integer roots of $P(x)$ must factor the constant term, 6, of $P(x)$. The only possible candidates for integer roots of $P(x)$ are $\pm 1$, $\pm 2$, $\pm 3$, $\pm 6$. Try $x = 1$. Because $P(1) = 0$, $x = 1$ is a root of $P(x)$ and $x - 1$ must be a factor of $P(x)$. By long division,

$$P(x) = (x - 1)(x^2 + 5x + 6) = (x - 1)(x + 2)(x + 3)$$

The degree of the numerator $18 - 2x - 4x^2$ is 2, which is smaller than the degree of the denominator, which is 3. So the integrand must be expressible in the form

$$\frac{18 - 2x - 4x^2}{(x-1)(x+2)(x+3)} = \frac{a}{x-1} + \frac{b}{x+2} + \frac{c}{x+3}$$

The easy way to find $a$ is to multiply both sides of this equation by $x - 1$

$$a + (x - 1) \frac{b}{x+2} + (x - 1) \frac{c}{x+3} = \frac{18 - 2x - 4x^2}{(x+2)(x+3)}$$

and then set $x = 1$

$$a = \frac{18 - 2x - 4x^2}{(x+2)(x+3)} \bigg|_{x=1} = \frac{18 - 2 - 4}{3 \times 4} = 1$$

Similarly,

$$b = \frac{18 - 2x - 4x^2}{(x-1)(x+3)} \bigg|_{x=-2} = \frac{18 + 4 - 16}{(-3) \times 1} = -2$$

$$c = \frac{18 - 2x - 4x^2}{(x-1)(x+2)} \bigg|_{x=-3} = \frac{18 + 6 - 36}{(-4) \times (-1)} = -3$$

To check this, put everything back over the common denominator $(x-1)(x+2)(x+3)$

$$\frac{1}{x-1} + \frac{-2}{x+2} + \frac{-3}{x+3} = \frac{(x+2)(x+3)-2(x-1)(x+3)-3(x-1)(x+2)}{(x-1)(x+2)(x+3)} = \frac{(1-2-3)x^2+(5-4-3)x+(6+6+6)}{(x-1)(x+2)(x+3)}$$

This is the same as the original integrand. So

$$\int \frac{18 - 2x - 4x^2}{x^3 + 4x^2 + x - 6} \, dx = \int \left[ \frac{1}{x-1} + \frac{-2}{x+2} + \frac{-3}{x+3} \right] \, dx = \ln |x - 1| - 2 \ln |x + 2| - 3 \ln |x + 3| + C'$$

b) Substitute $t = x + 1$, $dt = dx$.

$$\int \frac{x^3}{(x+1)^3} \, dx = \int \frac{(t-1)^3}{t^3} \, dt = \int \frac{t^3 - 3t^2 + 3t - 1}{t^3} \, dt = \int \left( 1 - 3t^{-1} + 3t^{-2} - t^{-3} \right) \, dt$$

$$= t - 3 \ln |t| - 3 \frac{1}{t} + \frac{1}{2t^2} + C = x - 3 \ln |x + 1| - 3 \frac{1}{x+1} + \frac{1}{2(x+1)^2} + C'$$

with $C' = C - 1$.

c) Trick question! The numerator is, aside from a factor of 2, the derivative of the denominator. So make the change of variables $u = x^4 - x^2 + 1$, $du = 4x^3 - 2x$.

$$\int \frac{2x^3 - x}{x^4 - x^2 + 1} \, dx = \int \frac{du/2}{u} = \frac{1}{2} \ln |u| + c = \frac{1}{2} \ln |x^4 - x^2 + 1| + c$$

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d) **Solution 1** Sub in \( x = \tan \theta \). Then \( dx = \sec^2 \theta \, d\theta \) and \( \sqrt{x^2 + 1} = \sqrt{\tan^2 \theta + 1} = \sec \theta \). When \( x = 0 \), \( \tan \theta = 0 \) so \( \theta = 0 \). When \( x = 1 \), \( \tan \theta = 1 \), so \( \theta = \frac{\pi}{4} \).

\[
\int_{0}^{1} \sqrt{x^2 + 1} \, dx = \int_{0}^{\pi/4} \sec \theta \, \sec^2 \theta \, d\theta = \int_{0}^{\pi/4} \sec^3 \theta \, d\theta
\]

\[
= \frac{1}{2} \left[ \sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \right]_{0}^{\pi/4}
\]

\[
= \frac{1}{2} \left[ \sqrt{2} + \ln |\sqrt{2} + 1| \right] - \frac{1}{2} \left[ 1 \times 0 + \ln |1 + 0| \right]
\]

\[
= \frac{1}{2} \left[ \sqrt{2} + \ln (\sqrt{2} + 1) \right]
\]

d) **Solution 2** Sub in \( x = \frac{1}{2}(e^y - e^{-y}) \). Then

\[
dx = \frac{1}{2}(e^y + e^{-y}) \, dy
\]

\[
\sqrt{x^2 + 1} = \left[ \frac{1}{4}(e^y - e^{-y})^2 + 1 \right]^{1/2} = \left[ \frac{1}{4}(e^{2y} - 2 + e^{-2y}) + 1 \right]^{1/2}
\]

\[
= \left[ \frac{1}{4}(e^{2y} + 2 + e^{-2y}) \right]^{1/2} = \left[ \frac{1}{4}(e^y + e^{-y})^2 \right]^{1/2}
\]

\[
= \frac{1}{2}(e^y + e^{-y})
\]

\[
\implies \sqrt{x^2 + 1} \, dx = \frac{1}{2}(e^y + e^{-y})(e^y - e^{-y}) \, dy = \frac{1}{4}(e^{2y} + 2 + e^{-2y}) \, dy
\]

\[
\implies \int \sqrt{x^2 + 1} \, dx = \frac{1}{4} \int (e^{2y} + 2 + e^{-2y}) \, dy = \frac{1}{8} e^{2y} + \frac{1}{2} y - \frac{1}{8} e^{-2y} + C
\]

When \( x = 0 \), \( e^y - e^{-y} = 0 \), so \( e^y = e^{-y} \) so \( e^{2y} = 1 \) so \( y = 0 \). When \( x = 1 \), \( e^y - e^{-y} = 2 \), so \( e^{2y} - 1 = 2e^y \) so \( e^{2y} - 2e^y - 1 = 0 \). View this as a quadratic equation in \( z = e^y \). The solutions to \( z^2 - 2z - 1 = 0 \) are \( z = \frac{1}{2}(2 \pm \sqrt{4 + 4}) = 1 \pm \sqrt{2} \). Since \( e^y > 0 \), only the solution \( z = \sqrt{2} + 1 \) is allowed. So when \( x = 1 \), \( e^y = \sqrt{2} + 1 \) and \( y = \ln (\sqrt{2} + 1) \).

\[
\int_{0}^{1} \sqrt{x^2 + 1} \, dx = \left[ \frac{1}{8} e^{2y} + \frac{1}{2} y - \frac{1}{8} e^{-2y} \right]_{0}^{\ln(\sqrt{2} + 1)} = \frac{1}{8} (\sqrt{2} + 1)^2 - \frac{1}{8} (\sqrt{2} + 1)^{-2} + \frac{1}{2} \ln (\sqrt{2} + 1)
\]

To simplify this use \( \frac{1}{\sqrt{2} + 1} = \frac{\sqrt{2} - 1}{(\sqrt{2} + 1)(\sqrt{2} - 1)} = \sqrt{2} - 1 \).

\[
\int_{0}^{1} \sqrt{x^2 + 1} \, dx = \frac{1}{8} (\sqrt{2} + 1)^2 - \frac{1}{8} (\sqrt{2} - 1)^{-2} + \frac{1}{2} \ln (\sqrt{2} + 1) = \frac{1}{2} \sqrt{2} + \frac{1}{2} \ln (\sqrt{2} + 1)
\]

32) Evaluate

a) \( \int \frac{e^x + 1}{e^x - 1} \, dx \)  

b) \( \int \frac{dx}{e^x + 1} \)  

c) \( \int \frac{1 + \sqrt{x}}{1 + \sqrt{x}} \, dx \)  

d) \( \int \sqrt{e^{2t} - 9} \, dt \)
Solution. a) Make the substitution \( y = e^x \), \( dy = e^x \, dx = y \, dx \).

\[
\int \frac{e^x + 1}{e^x - 1} \, dx = \int \frac{y^2 + 1}{y - 1} \, dy = \int \left[ \frac{2}{y - 1} - \frac{1}{y} \right] \, dy = 2 \ln |y - 1| - \ln |y| + C \\
= 2 \ln |e^x - 1| - \ln |e^x| + C \\
= 2 \ln |e^{x/2} - e^{-x/2}| + C = \ln |e^x - 2 + e^{-x}| + C
\]

b) Make the substitution \( y = e^x \), \( dy = e^x \, dx = y \, dx \).

\[
\int \frac{dx}{e^x + 1} = \int \frac{1}{y + 1} \, dy = \int \left[ \frac{1}{y} - \frac{1}{y + 1} \right] \, dy = \ln |y| - \ln |y + 1| + C \\
= \ln e^x - \ln (e^x + 1) + C = - \ln \left( 1 + e^{-x} \right) + C
\]

c) Let \( x = u^6 \).

\[
\int \frac{dx}{1 + u^2} = \int \frac{1}{1 + u^2} 6u^5 \, du \\
\text{By long division} \\
\frac{u^8 + u^5}{1 + u^2} = u^6 - u^4 + u^3 + u^2 - u - 1 + \frac{u^6}{1 + u^2}
\]

Hence

\[
\int \frac{1 + \sqrt{x}}{1 + \sqrt{x}} \, dx = 6 \int \left[ u^6 - u^4 + u^3 + u^2 - u - 1 + \frac{u^6}{1 + u^2} \right] \, du \\
= 6 \left[ \frac{u^7}{7} - \frac{u^5}{5} + \frac{u^4}{4} - \frac{u^3}{3} - \frac{u^2}{2} - u + \frac{1}{2} \ln (1 + u^2) + \tan^{-1} u \right] + C \\
= \frac{6}{7} x^{7/6} - \frac{6}{5} x^{5/6} + \frac{3}{2} x^{2/3} + 2x^{1/2} - 3x^{1/3} - 6x^{1/6} + 3 \ln (1 + x^{1/3}) + 6 \tan^{-1} x^{1/6} + C
\]

a) Make the substitution \( y = e^t \), \( dy = e^t \, dt = y \, dt \).

\[
\int \sqrt{e^{2t} - 9} \, dt = \int \sqrt{y^2 - 9} \, \frac{du}{y}
\]

Now substitute \( y = 3 \sec u \), \( dy = 3 \sec u \tan u \, du \). Since \( \sqrt{9 \sec^2 u - 9} = \sqrt{9 \tan^2 u} = 3 \tan u \),

\[
\int \sqrt{e^{2t} - 9} \, dt = \int \sqrt{9 \sec^2 u - 9} \, \frac{3 \sec u \tan u \, du}{3 \sec u} = 3 \int \tan^2 u \, du = 3 \int (\sec^2 u - 1) \, du \\
= 3 \left[ \tan u - u \right] + C = 3 \left[ \sqrt{\sec^2 u - 1} - u \right] + C \\
= 3 \left[ \sqrt{\left( \frac{y}{3} \right)^2 - 1} - \sec^{-1} \frac{y}{3} \right] + C = \sqrt{e^{2t} - 9} - 3 \sec^{-1} \frac{e^t}{3} + C
\]

33) The circle \( x^2 + (y - 1)^2 = 2 \) and the parabola \( y = \frac{1}{2} (x^2 - 1) \) are tangent to each other.
a) Carefully plot these two curves and determine the vertex of the parabola and the points of tangency.
b) Find the area of the crescent shaped region bounded by these curves.

**Solution.** a) The parabola is symmetric about the \(x\)-axis. It hits the \(x\)-axis at \(y = -1/2\). So the vertex is \((0, -1/2)\). Any point of tangency must also be a point of intersection. At a point of intersection, 
\[x^2 = 2 - (y - 1)^2\] and 
\[x^2 = 2y + 1\] so 
\[2y + 1 = 2 - (y - 1)^2\] or 
\[y^2 = 0\] or 
\[y = 0\]. The corresponding values of \(x\) are 
\[x = \pm 1\]. We are told that there is at least one point of tangency. By symmetry about the \(x\)-axis \(\pm (1, 0)\) are both points of tangency.
b) The bottom half of the circle has equation \(1 - \sqrt{2 - x^2}\). So the area is 
\[
\int_{-1}^{1} \left[1 - \sqrt{2 - x^2} - \frac{1}{2}(x^2 - 1)\right] dx = \int_{0}^{1} \left[3 - x^2 - 2\sqrt{2 - x^2}\right] dx
\]
\[= 3x - \frac{x^3}{3}\bigg|_{0}^{1} - 2 \int_{0}^{1} \sqrt{2 - x^2} dx = \frac{8}{3} - 2 \int_{0}^{1} \sqrt{2 - x^2} dx
\]
For the remaining integral, sub in \(x = \sqrt{2} \sin t\), 
\[dx = \sqrt{2} \cos t\, dt\]. When \(x = 0\), \(t = 0\) and when \(x = 1\), \(t = \frac{\pi}{4}\). The area is 
\[\frac{8}{3} - 2 \int_{0}^{\pi/4} \sqrt{2 - \sqrt{2} \cos t} \cos t\, dt = \frac{8}{3} - 4 \int_{0}^{\pi/4} \cos^2 t\, dt
\]
\[= \frac{8}{3} - 4 \left[1 + \cos 2t\right] dt = \frac{8}{3} - 2 \left[t + \frac{\sin 2t}{2}\right]_{0}^{\pi/4}
\]
\[= \frac{8}{3} - 2 \left[\frac{\pi}{4} + \frac{1}{2}\right] = \frac{5}{3} - \frac{\pi}{2}
\]

34) Let \(R\) be the region in the \(xy\)-plane bounded by \(y = \sqrt{\frac{x^2+1}{x}},\ y = 0,\ x = 1\ and \ x = \sqrt{3}\).
a) Find the volume of the solid obtained by rotating \(R\) about the \(x\)-axis.
b) Find the volume of the solid obtained by rotating \(R\) about the \(y\)-axis.

**Solution.** a)
\[
Vol = \int_{1}^{\sqrt{3}} \pi \left(\sqrt{\frac{x^2+1}{x}}\right)^2 dx = \pi \int_{1}^{\sqrt{3}} \left(1 + \frac{1}{x^2}\right) dx = \pi \left[x - \frac{1}{x}\right]_{1}^{\sqrt{3}} = \pi \left[\sqrt{3} - \frac{1}{\sqrt{3}}\right]
\]
b) By cylindrical shells,
\[
Vol = \int_{1}^{\sqrt{3}} 2\pi x \left(\sqrt{\frac{x^2+1}{x}}\right) dx = 2\pi \int_{1}^{\sqrt{3}} \sqrt{x^2 + 1}\, dx
\]
Sub in $x = \tan t$, $dx = \sec^2 t \, dt$, $1 = \tan \frac{\pi}{4}$, $\sqrt{3} = \tan \frac{\pi}{3}$.

$$\text{Vol} = 2\pi \int_{\pi/4}^{\pi/3} \sqrt{\tan^2 t + 1} \sec^2 t \, dt = 2\pi \int_{\pi/4}^{\pi/3} \sec^3 t \, dt = 2\pi \int_{\pi/4}^{\pi/3} \frac{\cos t}{\cos^4 t} \, dt$$

Sub in $u = \sin t$, $du = \cos t \, dt$, $\cos^4 t = (1 - \sin^2 t)^2 = (1 - u^2)^2$, $\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$$\text{Vol} = 2\pi \int_{\sqrt{3}/2}^{\sqrt{3}/2} \frac{1}{(1-u^2)^2} \, du = 2\pi \int_{1/\sqrt{2}}^{1/\sqrt{2}} \left[ \frac{1}{1-u} + \frac{2}{1+1+u} \right]^2 \, du$$

$$= \frac{\pi}{2} \int_{1/\sqrt{2}}^{1/\sqrt{2}} \left[ \frac{1}{(1-u)^2} + \frac{2}{(1-u)(1+u)} + \frac{1}{(1+u)^2} \right] \, du$$

$$= \frac{\pi}{2} \int_{1/\sqrt{2}}^{1/\sqrt{2}} \left[ \frac{1}{1-u} - \ln |1-u| + \ln |1+u| - \frac{1}{1+u} \right] \left[ \frac{1}{1-u} + \ln |1-u| + \ln |1+u| - \frac{1}{1+u} \right] \sqrt{3}/2$$

$$= \frac{\pi}{2} \left[ \frac{2u}{1-u^2} + \ln \left| \frac{1+u}{1-u} \right| \right] \sqrt{3}/2 = \frac{\pi}{2} \left[ \frac{\sqrt{3}}{1/4} + \ln \frac{2+\sqrt{3}}{2-\sqrt{3}} - \frac{\sqrt{2}}{\sqrt{2}-1} - \ln \frac{\sqrt{2}+1}{\sqrt{2}-1} \right]$$

$$= 2\pi \left[ \sqrt{3} + \frac{1}{4} \ln \frac{2+\sqrt{3}}{2-\sqrt{3}} - \frac{\sqrt{2}}{2} - \frac{1}{4} \ln \frac{\sqrt{2}+1}{\sqrt{2}-1} - \frac{\sqrt{2}}{2} - \frac{1}{2} \ln(\sqrt{2} + 1) \right]$$

35) A swimming pool has the shape shown in the figure below. The vertical cross-sections of the pool are squares. The distances in feet across the pool are given in the figure at 2 foot intervals along the sixteen foot length of the pool.

a) Denote by $f(x)$ the width of the pool at a distance $x$ from the left hand end. Express the volume of the pool as an integral in terms of $f$.

b) Find an approximate value for the integral of part a.
Solution. a) The volume of the part of the pool with $x$-coordinate running from $x$ to $x + dx$ is $f(x)^2 \, dx$. So the total volume is

$$V = \int_0^{16} f(x)^2 \, dx$$

b) Divide the domain of integration into 8 intervals each of length $\Delta x = 2$. Denote by $x_i^*$ the right hand end point of interval number $i$. Thus $x_1^* = 2$, $x_2^* = 4$, $\ldots$, $x_8^* = 16$. The corresponding values of $f$ are $f(x_1^*) = f(2) = 10$, $f(x_2^*) = f(4) = 12$, $f(x_3^*) = 10$, $f(x_4^*) = 8$, $f(x_5^*) = 6$, $f(x_6^*) = 8$, $f(x^*_7) = 10$ and $f(x_8^*) = 0$. Then

$$V = \int_0^{16} f(x)^2 \, dx$$

$$\approx \Delta x \left[ f(x_1^*)^2 + f(x_2^*)^2 + f(x_3^*)^2 + f(x_4^*)^2 + f(x_5^*)^2 + f(x_6^*)^2 + f(x_7^*)^2 + f(x_8^*)^2 \right]$$

$$= 2 \left[ 10^2 + 12^2 + 10^2 + 8^2 + 6^2 + 8^2 + 10^2 + 0^2 \right] = 1216 \text{ ft}^3$$

36) Consider the function $F(x) = \int_0^x t^2 e^{-t^2} \, dt - \int_{-x}^0 2t^4 e^{-t^4} \, dt$. Find the values of $x$ for which $F(x)$ takes its maximum and minimum values on $0 \leq x \leq 1 + a^2$. The number $a$ has the property that $F(1 + a^2) > 0$.

Solution. We first compute $F'(x)$. To do so we write

$$F(x) = G(x^2) + H(-x) \quad \text{with} \quad G(y) = \int_0^y t^2 e^{-t^2} \, dt, \quad H(y) = \int_0^y 2t^4 e^{-t^4} \, dt$$

By the Fundamental Theorem of Calculus,

$$G'(y) = y^2 e^{-y^2} \quad H'(y) = 2y^4 e^{-y^4}$$

Hence, by the chain rule

$$F'(x) = 2xG'(x^2) - H'(-x) = 2x^5 e^{-x^4} - 2x^4 e^{-x^4} = 2x^4(x - 1)e^{-x^4}$$

Observe that $F'(x) < 0$ for $x < 1$ and $F'(x) > 0$ for $x > 1$. Hence $F(x)$ is decreasing for $x < 1$ and increasing for $x > 1$ and $F(x)$ must take its minimum value when $x = 1$. It may take its maximum value at $x = 0$ or $x = 1 + a^2$. But $F(0) = 0$ and we are told that $F(1 + a^2) > 0$. Hence $F(x)$ takes its maximum value when $x = 1 + a^2$.

37) Recall that, when $n$ is odd, it is easy to integrate $\int \sin^m x \cos^n x \, dx$ using the substitution $y = \sin x$. The integral $\int \sec x \, dx$ is of the form $\int \sin^m x \cos^n x \, dx$ with $m = 0$ and $n = -1$, which is odd. Integrate $\int \sec x \, dx$ by substituting $y = \sin x$. 

27
Solution. Substitute \( y = \sin x \), \( dy = \cos x \, dx \). Then

\[
\int \sec x \, dx = \int \frac{\cos x \, dx}{\cos^2 x} = \int \frac{\cos x \, dx}{1 - \sin^2 x} = \int \frac{dy}{1 - y^2} = -\int \frac{dy}{(y-1)(y+1)}
\]

\[
= -\int \left[ \frac{1/2}{y-1} - \frac{1/2}{y+1} \right] \, dy = -\frac{1}{2} \ln |y - 1| + \frac{1}{2} \ln |y + 1| + C = \frac{1}{2} \ln \frac{\sin x + 1}{\sin x - 1} + C
\]

To make the connection to the indefinite integral

\[
\int \sec x \, dx = \ln |\sec x + \tan x| + C,
\]

observe that

\[
\frac{\sin x + 1}{\sin x - 1} = \frac{(\sin x + 1)^2}{\sin^2 x - 1} = -\left( \frac{\sin x + 1}{\cos x} \right)^2 = -\left( \tan x + \sec x \right)^2
\]

Hence

\[
\int \sec x \, dx = \frac{1}{2} \ln \left| \frac{\sin x + 1}{\sin x - 1} \right| + C = \frac{1}{2} \ln \left| \tan x + \sec x \right|^2 + C = \boxed{\ln \left| \tan x + \sec x \right| + C}
\]

38) Evaluate, using the substitution \( x = \tan \frac{\theta}{2} \).

a) \( \int \frac{d\theta}{2 + \sin \theta} \)

b) \( \int_0^{\pi/2} \frac{d\theta}{1 + \sin \theta + \cos \theta} \)

c) \( \int \frac{d\theta}{3 + 2 \cos \theta} \)

Solution. If \( x = \tan \frac{\theta}{2} \), then, from the right-angled triangle with sides \( x \) and 1 and hypotenuse \( \sqrt{1 + x^2} \), \( \sin \frac{\theta}{2} = \frac{x}{\sqrt{1 + x^2}} \) and \( \cos \frac{\theta}{2} = \frac{1}{\sqrt{1 + x^2}} \) so that

\[
\sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = \frac{2x}{1 + x^2}
\]

\[
\cos \theta = 2 \cos^2 \frac{\theta}{2} - 1 = 2 \frac{1}{1 + x^2} - 1 = \frac{1 - x^2}{1 + x^2}
\]

\[
d\theta = 2 \cos^2 \frac{\theta}{2} \, dx = \frac{2}{1 + x^2} \, dx
\]

a)

\[
\int \frac{d\theta}{2 + \sin \theta} = \int \frac{1}{2 + \frac{2x}{1 + x^2}} \, dx = \int \frac{1}{1 + x + x^2} \, dx = \int \frac{1}{(x + 1)^2 + \frac{3}{4}} \, dx
\]

Now sub \( x + \frac{1}{2} = \frac{\sqrt{3}}{2} y \), \( dx = \frac{\sqrt{3}}{2} \, dy \)

\[
\int \frac{1}{(x + \frac{1}{2})^2 + \frac{3}{4}} \, dx = \int \frac{1}{\frac{3}{4} y^2 + \frac{3}{4}} \, dy = \frac{2}{\sqrt{3}} \int \frac{1}{y^2 + 1} \, dy = \frac{2}{\sqrt{3}} \tan^{-1} \left( \frac{2 \sqrt{3} (x + \frac{1}{2})}{\sqrt{3}} \right) + C
\]

\[
= \frac{2}{\sqrt{3}} \tan^{-1} \left( \frac{2 \tan \frac{\theta}{2} + 1}{\sqrt{3}} \right) + C
\]

b)

\[
\int_0^{\pi/2} \frac{d\theta}{1 + \sin \theta + \cos \theta} = \int_0^1 \frac{1}{1 + \frac{2x}{1 + x^2} + \frac{1 - x^2}{1 + x^2}} \, dx = \int_0^1 \frac{1}{1 + x^2 + 2x + 1 - x^2} \, dx = \int_0^1 \frac{1}{1 + x} \, dx = \ln(1 + x) \bigg|_0^1 = \boxed{\ln 2}
\]
\[
\int \frac{d\theta}{3 + 2\cos \theta} = \int \frac{1}{3 + 2\frac{x^2 - 1}{1 + x^2}} \frac{2}{1 + x^2} \, dx = \int \frac{2}{3 + 3x^2 + 2 - 2x^2} \, dx = 2 \int \frac{1}{5 + x^2} \, dx
\]

\[
= \frac{2}{\sqrt{5}} \tan^{-1} \frac{x}{\sqrt{5}} + C = \frac{2}{\sqrt{5}} \tan^{-1} \left( \frac{\tan \frac{\theta}{2}}{\sqrt{5}} \right) + C
\]

39) Use \( \cos \theta = \frac{1}{2} (e^{i\theta} + e^{-i\theta}) \) and \( \sin \theta = \frac{1}{2i} (e^{i\theta} - e^{-i\theta}) \) to find identities for \( \sin(3\theta) \) and \( \cos(3\theta) \) in terms of \( \sin \theta \) and \( \cos \theta \).

**Solution.** Note that

\[
\sin^3 \theta = -\frac{1}{8} (e^{i\theta} - e^{-i\theta})^3
\]

\[
= -\frac{1}{8} (e^{3i\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-3i\theta})
\]

\[
= \frac{3}{4} \frac{1}{2i} (e^{i\theta} - e^{-i\theta}) - \frac{1}{4} \frac{1}{2i} (e^{i3\theta} - e^{-i3\theta})
\]

\[
= \frac{3}{4} \sin \theta - \frac{1}{4} \sin(3\theta)
\]

and

\[
\cos^3 \theta = \frac{1}{8} (e^{i\theta} + e^{-i\theta})^3
\]

\[
= \frac{1}{8} (e^{3i\theta} + 3e^{i\theta} + 3e^{-i\theta} + e^{-3i\theta})
\]

\[
= \frac{3}{4} \frac{1}{2} (e^{i\theta} + e^{-i\theta}) + \frac{1}{4} \frac{1}{2i} (e^{i3\theta} + e^{-i3\theta})
\]

\[
= \frac{3}{4} \cos \theta + \frac{1}{4} \cos(3\theta)
\]

Hence \( \sin(3\theta) = 3 \sin \theta - 4 \sin^3 \theta \) and \( \cos(3\theta) = 4 \cos^3 \theta - 3 \cos \theta \)

40) For each natural number \( n \) and each nonzero complex number \( a \), the equation \( z^n = a \) has precisely \( n \) distinct solutions, called the \( n \)th roots of \( a \).

a) Find the three cube roots of \(-1\).

b) Show that the sum of the \( n \) \( n \)th roots of \( 1 \) is zero.

**Solution.** a) Note that

\[
(e^{i\pi/3})^3 = e^{i\pi} = -1
\]

\[
(e^{i3\pi/3})^3 = e^{i3\pi} = e^{i\pi} = -1 \text{ since } e^{i2\pi} = 1
\]

\[
(e^{i5\pi/3})^3 = e^{i5\pi} = e^{i\pi} = -1 \text{ since } e^{i4\pi} = 1
\]

Thus \( e^{i\pi/3} = \frac{1}{2} + \frac{\sqrt{3}}{2} i \), \( e^{i3\pi/3} = -1 \) and \( e^{i5\pi/3} = e^{-i\pi/3} = \frac{1}{2} - \frac{\sqrt{3}}{2} i \) are three distinct cube roots of \(-1\).

b) For all integers \( k \), \( e^{2k\pi i} = 1 \). Thus for every integer \( k \)

\[
(e^{i2k\pi/n})^n = e^{i2k\pi} = 1
\]
and $e^{i2k\pi/n}$ is an $n^{th}$ root of 1. The $n$ complex numbers

$$1 = e^{i0\pi/n}, e^{i2\pi/n}, e^{i2\times2\pi/n}, e^{i3\times2\pi/n}, \ldots, e^{i(n-1)\times2\pi/n}$$

are all different and so are the $n n^{th}$ roots of 1. Write $z_p = e^{i2\pi/n}$. The $n n^{th}$ roots of 1 are 1, $z_p$, $z_p^2$, $z_p^3$, $\ldots$, $z_p^{n-1}$. Using the formula for the partial sum of a geometric series with ratio $z_p$

$$1 + z_p + z_p^2 + \cdots + z_p^{n-1} = \frac{1-z_p^n}{1-z_p}$$

The numerator vanishes because $z_p$ is an $n^{th}$ root of 1.

41) Evaluate the given integrals, using complex numbers.

a) $\int_0^1 \frac{2x^2-2}{(x^2+1)^2} \, dx$

b) $\int_0^{\pi/2} \sin^4 \theta \, d\theta$

**Solution.** a) We use partial fractions.

$$\frac{2x^2-2}{(x+i)^2(x-i)^2} = \frac{a}{x+i} + \frac{b}{(x+i)^2} + \frac{c}{x-i} + \frac{d}{(x-i)^2}$$

For this to be true for all $x$, we need $a(x+i)(x-i)^2 + b(x-i)^2 + c(x-i)(x+i)^2 + d(x+i)^2 = 2x^2 - 2$ for all $x$. Setting $x = i$ gives $(2i)^2d = -4$ or $d = 1$. Setting $x = -i$ gives $(-2i)^2b = -4$ or $b = 1$. Subbing $b = d = 1$ back in and using $b(x-i)^2 + d(x+i)^2 = (x-i)^2 + (x+i)^2 = 2x^2 - 2$ gives $a(x+i)(x-i)^2 + c(x-i)(x+i)^2 = 0$ or $a = c = 0$. Hence

$$\int_0^1 \frac{2x^2-2}{(x^2+1)^2} \, dx = \left[ -\frac{1}{x+i} - \frac{1}{x-i} \right]_0^1 = \left[ -\frac{x-i+x+i}{(x+i)(x-i)} \right]_0^1 = -\frac{2x}{x^2+1} \bigg|_0^1 = -1$$

b) Since $\sin \theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$,

$$\int_0^{\pi/2} \sin^4 \theta \, d\theta = \int_0^{\pi/2} \left( e^{i\theta} - e^{-i\theta} \right)^4 d\theta$$

$$= \int_0^{\pi/2} (e^{4i\theta} - 4e^{2i\theta} + 6 - 4e^{-2i\theta} + e^{-4i\theta})^4 d\theta$$

$$= \frac{1}{16} \left[ e^{4i\theta} - 2e^{2i\theta} + 6\theta + \frac{2}{i}e^{-2i\theta} - \frac{1}{4}e^{-4i\theta} \right]_0^{\pi/2}$$

$$= \frac{1}{16} \left[ \frac{1}{2} \sin(4\theta) - 4\sin(2\theta) + 6\theta \right]_0^{\pi/2} = \frac{1}{16} \frac{6\pi}{2} = \frac{3}{16}\pi$$

42) Solve

a) $\frac{dx}{dt} = 1 + t - x - tx$

b) $\frac{dy}{dt} = \frac{ty^3}{t^2+1}$, $y(2) = 2$

c) $\frac{dy}{dx} = y^2$
Solution. a)

\[ \frac{dx}{dt} = 1 + t - x - tx = (1 + t)(1 - x) \Rightarrow \frac{dx}{x-1} = (1 + t) \, dt \Rightarrow -\ln|x-1| = t + \frac{t^2}{2} + D \]

\[ \Rightarrow |x - 1| = e^D e^{-t - t^2/2} \Rightarrow x - 1 = \pm e^D e^{-t - t^2/2} \Rightarrow x = 1 + Ce^{-t - t^2/2} \]

b)

\[ \frac{dy}{dt} = \frac{ty + 3t}{t^2 + 1} \Rightarrow \frac{dy}{y+3} = \frac{t}{t^2 + 1} \Rightarrow \ln|y + 3| = \frac{1}{2} \ln(t^2 + 1) + D \]

\[ \Rightarrow |y + 3| = e^D \sqrt{t^2 + 1} \Rightarrow y = -3 + C\sqrt{t^2 + 1} \text{ where } C = \pm e^D \]

To satisfy \( y(2) = 2 \), we need \( 2 = -3 + \sqrt{5} \) or \( C = \sqrt{5} \), so \( y = -3 + \sqrt{5} t^2 + 5 \).

c) When \( y \neq 0 \),

\[ \frac{dy}{dx} = y^2 \quad \Rightarrow \quad \frac{dy}{y^2} = dx \quad \Rightarrow \quad y^{-1} = x + c \quad \Rightarrow \quad y = \frac{1}{x + c} \]

When \( y = 0 \), this computation breaks down because \( \frac{dy}{y^2} \) contains a division by 0. We can check if the function \( y(x) = 0 \) satisfies the differential equation by just subbing it in:

\[ y(x) = 0 \quad \Rightarrow \quad y'(x) = 0, \quad y(x)^2 = 0 \quad \Rightarrow \quad y'(x) = y(x)^2 \]

So \( y(x) = 0 \) is a solution and the full solution is

\[ y(x) = 0 \text{ or } y(x) = -\frac{1}{x + c}, \text{ for any constant } c \]

43) Find an equation for the curve that passes through the point \((1, 1)\) and whose slope at \((x, y)\) is \( \frac{y^2}{x^3} \).

**Solution.** We need to find a function \( y(x) \) obeying \( \frac{dy}{dx} = \frac{y^2}{x^3} \) and \( y(1) = 1 \).

\[ \frac{dy}{dx} = \frac{y^2}{x^3} \quad \Rightarrow \quad \frac{dy}{y^2} = \frac{dx}{x^3} \quad \Rightarrow \quad y^{-1} = x^{-2} + c \]

To satsify \( y(1) = 1 \), we need \( -1 = -\frac{1}{2} + c \) or \( c = -\frac{1}{2} \). So \( -\frac{1}{y} = -\frac{1}{2x^2} - \frac{1}{2} = -\frac{1 + x^2}{2x^2} \) or \( y = \frac{2x^2}{x^2 + 1} \).

44) In 1986, the population of the world was 5 billion and was increasing at a rate of 2% per year. Using the logistic growth model with an assumed maximum population of 100 billion, predict the population of the world in the years 2000, 2100 and 2500.

**Solution.** Let \( y(t) \) be the population of the world, in billions of people, at time 1986 + \( t \). The logistic growth model assumes

\[ y' = ky(M - y) \]
We know that, if at time zero the population is below \( M \), then as time increases the population increases, approaching the limit \( M \) as \( t \) tends to infinity. So in this problem \( M \) is the maximum population. That is, \( M = 100 \). We are also told that, at time zero, the percentage rate of change of population, \( 100 \frac{dy}{y} \), is 2, so that, at time zero, \( \frac{dy}{y} = 0.02 \).

But, from the differential equation, \( \frac{dy}{y} = k(M - y) \). Hence at time zero, \( 0.02 = k(100 - 5) \), so that \( k = \frac{2}{9500} \). We now know \( k \) and \( M \) and can solve the differential equation

\[
\frac{dy}{dt} = ky(M - y) \Rightarrow \frac{dy}{y(M - y)} = k dt \Rightarrow \int \frac{1}{M} \left[ \frac{1}{y} - \frac{1}{y - M} \right] dy = kt + C
\]

\[
\Rightarrow \frac{1}{M} [\ln |y| - \ln |y - M|] = kt + C \Rightarrow \ln \left| \frac{y}{y - M} \right| = kMt + CM \Rightarrow \left| \frac{y}{y - M} \right| = De^{kMt}
\]

with \( D = e^{CM} \). We know that \( y \) remains between 0 and \( M \), so that \( \left| \frac{y}{y - M} \right| = \frac{y}{M - y} = De^{kMt} \). We are given that at \( t = 0 \), \( y = 5 \). Subbing this, and the values of \( M \) and \( k \), in

\[
\frac{5}{100 - 5} = De^{0} \Rightarrow D = \frac{5}{95} \Rightarrow \frac{y}{100 - y} = \frac{5}{95} e^{2t/95} \Rightarrow y = (100 - y) \frac{5}{95} e^{2t/95}
\]

\[
\Rightarrow 95y = (500 - 5y) e^{2t/95} \Rightarrow y = \frac{500 e^{2t/95}}{95 + 5 e^{2t/95}} = \frac{100 e^{2t/95}}{1 + 19 e^{-2t/95}} = \frac{100}{1 + 19 e^{-2t/95}} \approx 6.6 \text{ billion}
\]

In the year 2000, \( t = 14 \) and \( y = \frac{100}{1 + 19 e^{-28/95}} \approx 6.6 \text{ billion} \). In the year 2100, \( t = 114 \) and \( y = \frac{100}{1 + 19 e^{-228/95}} \approx 36.7 \text{ billion} \). In the year 2200, \( t = 514 \) and \( y = \frac{100}{1 + 19 e^{-1028/95}} \approx 100 \text{ billion} \).

45) When a raindrop falls it increases in size so that its mass \( m(t) \), is a function of time \( t \). The rate of growth of mass is \( km(t) \) for some positive constant \( k \). According to Newton’s law of motion, \( \frac{d}{dt}(mv) = gm \), where \( v \) is the speed of the raindrop and \( g \) is the acceleration due to gravity. Find the terminal velocity \( \lim_{t \to \infty} v(t) \) of a raindrop.

Solution. First we determine \( m(t) \).

\[
\frac{dm}{dt} = km \quad \Rightarrow \quad \frac{dm}{m} = k dt \quad \Rightarrow \quad \ln m = kt + c \quad \Rightarrow \quad m = de^{kt} \quad \text{where} \quad d = e^{c}
\]

Define \( y(t) = m(t)v(t) \). Then

\[
\frac{dy}{dt} = gm(t) = gde^{kt} \quad \Rightarrow \quad dy = gde^{kt} dt \quad \Rightarrow \quad y = gd\frac{e^{kt}}{k} + C
\]

Subbing back in \( y(t) = m(t)v(t) = de^{kt}v(t) \),

\[
y = gd\frac{e^{kt}}{k} + C \quad \Rightarrow \quad de^{kt}v(t) = gd\frac{e^{kt}}{k} + C \quad \Rightarrow \quad v(t) = \frac{q}{k} + \frac{C}{de^{kt}}
\]

Since \( k > 0 \), \( \lim_{t \to \infty} \frac{C}{e^{kt}} = 0 \) and \( \lim_{t \to \infty} v(t) = \frac{q}{k} \).
46) A population of rare South American monkeys has been found to have a population growth rate which decreases with time. Specifically, the population size, \( y(t) \), satisfies the differential equation

\[
y'(t) = \frac{y}{(t+1)^2}
\]

a) Given that at time \( t = 0 \) the population is 100, solve for the population size \( y(t) \) after time \( t \).

b) In the limit \( t \to \infty \), what is the size of the population?

**Solution.**

\[
y'(t) = \frac{y}{(t+1)^2} \iff dy = \frac{y}{(t+1)^2} dt \iff \frac{dy}{y} = \frac{dt}{(t+1)^2} \iff \ln y = \frac{(t+1)^{-1}}{-1} + C
\]

where \( K = e^C \). At \( t = 0, y = 100 \). Subbing this in \( 100 = Ke^{-1} \), so \( K = 100e \) and \( y(t) = 100e^{-(t+1)^{-1}+1} \).

b) As \( t \to \infty \), \( \frac{1}{t+1} \) approaches 0. So

\[
\lim_{t \to \infty} y(t) = \lim_{t \to \infty} 100e^{-(t+1)^{-1}+1} = 100e^{0+1} = 100e
\]

47) Suppose that \( y(t) \) (measured in thousands of dollars – “kilobucks”) represents the balance in a savings account at time \( t \) years after the account was opened with an initial balance of 2 kilobucks. Interest is being paid into the account continuously at a nominal rate of 5% per annum; that is to say, interest is flowing into the account at a rate of \( \frac{5}{100}y(t) \) kilobucks per year at time \( t \). At the same time, the account is being continuously depleted by taxes and other charges at a rate \( \frac{y(t)^2}{1000} \) kilobucks per year. No other deposits or withdrawals are made on the account.

a) What is balance in the account at time \( t = 20 \) years?

b) How large can the balance in the account grow over time?

**Solution.**

a) The differential equation determining the time dependence of \( y(t) \) is

\[
\frac{dy}{dt} (t) = \frac{5}{100} y(t) - \frac{y(t)^2}{1000} \implies -1000 \frac{dy}{dt} = -50y + y^2 \implies \frac{1000}{y^2 - 50y} dy = -dt
\]

The integral of the left hand side is

\[
\int \frac{1000}{y^2 - 50y} dy = \int \frac{1000}{(y-50)y} dy = \int \left[ \frac{20}{y-50} - \frac{20}{y} \right] dy = 20 \ln |y - 50| - 20 \ln |y| + C
\]

Before continuing let’s discuss the sign of \( \frac{y-50}{y} \). We start with \( y = 2 \). The rate of change of \( y \) is determined by \( \frac{dy}{dt} (t) = \frac{5}{100} y(t) - \frac{y(t)^2}{1000} = \frac{1}{1000} y(50 - y) \). Note that the rate of change is positive for \( 0 \leq y \leq 50 \) and negative otherwise. So, as we start with \( y = 2, y \)}
increases at the beginning and will continue to do so as long as \( y \leq 50 \). But \( y \) can never get above 50, because \( \frac{dy}{dt} \) is negative, that is \( y \) is decreasing in time, for all \( y > 50 \). So \( y \) must remain between 0 and 50 and \( \left| \frac{y-50}{y} \right| = \frac{50-y}{y} \). Now back to the solution.

\[
\frac{dy}{dt}(t) = \frac{5}{100}y(t) - \frac{y(t)^2}{1000} \implies \frac{1000}{y^2-50y}dy = -dt \implies 20 \ln \frac{50-y}{y} + C = -t
\]

Dividing by 20 and then taking the exponential of both sides

\[
\frac{50-y}{y} = Ke^{-t/20}
\]

where \( K = e^{-C} \). At \( t = 0, y = 2 \), so \( K = \frac{50-2}{2} = 24 \).

\[
\frac{50-y}{y} = 24e^{-t/20} \implies 50 - y = 24ye^{-t/20} \implies (1 + 24e^{-t/20})y = 50
\]

\[
\implies y = \frac{50}{1 + 24e^{-t/20}} \quad \text{and} \quad y(20) \approx 5.087
\]

b) As \( t \) increases, \( 1 + 24e^{-t/20} \) decreases, approaching the limit 0 as \( t \) tends to infinity. Hence \( y \) increases over time, but remains below 50, approaching 50 as time tends to infinity.

48 a) Let \( T_n \) and \( M_n \) be the \( n \)-step Trapezoidal and Midpoint Rule approximations, respectively, for \( \int_0^1 \cos(x^2) \, dx \). Find \( T_4, T_8, M_4 \) and \( M_8 \).

b) Estimate the errors involved in these approximations.

**Solution.** a)

\[
T_4 = \frac{1}{4} \left[ \frac{1}{2} \cos 0 + \cos \left( \frac{1}{4} \right)^2 + \cos \left( \frac{2}{4} \right)^2 + \cos \left( \frac{3}{4} \right)^2 + \frac{1}{2} \cos 1 \right] = 0.895759
\]

\[
T_8 = \frac{1}{8} \left[ \frac{1}{2} \cos 0 + \cos \left( \frac{1}{8} \right)^2 + \cos \left( \frac{2}{8} \right)^2 + \cos \left( \frac{3}{8} \right)^2 + \cdots + \cos \left( \frac{7}{8} \right)^2 + \frac{1}{2} \cos 1 \right] = 0.902333
\]

\[
M_4 = \frac{1}{4} \left[ \cos \left( \frac{1}{8} \right)^2 + \cos \left( \frac{2}{8} \right)^2 + \cos \left( \frac{3}{8} \right)^2 + \cos \left( \frac{4}{8} \right)^2 \right] = 0.908907
\]

\[
M_8 = \frac{1}{8} \left[ \cos \left( \frac{1}{16} \right)^2 + \cos \left( \frac{3}{16} \right)^2 + \cos \left( \frac{5}{16} \right)^2 + \cdots + \cos \left( \frac{15}{16} \right)^2 \right] = 0.905620
\]

b)

\[
f(x) = \cos(x^2) \implies f'(x) = -2x \sin(x^2) \implies f''(x) = -2 \sin(x^2) - 4x^2 \cos(x^2)
\]

On the interval \( 0 \leq x \leq 1 \), \( |\sin(x^2)| \) is never bigger than 1 and \( |\cos(x^2)| \) is never bigger than 1 and \( x^2 \) is never bigger than 1, so \( |f''(x)| \) is never bigger than \( 2 \times 1 + 4 \times 1 = 6 \). Hence

\[
|E_{T_4}| \leq \frac{6(1-0)^3}{12 \times 4^2} \leq 0.0313
\]

\[
|E_{T_8}| \leq \frac{6(1-0)^3}{12 \times 8^2} \leq 0.00782
\]

\[
|E_{M_4}| \leq \frac{6(1-0)^3}{24 \times 4^2} \leq 0.0157
\]

\[
|E_{M_8}| \leq \frac{6(1-0)^3}{24 \times 8^2} \leq 0.00391
\]
49) How large should \( n \) be to ensure that the Simpson’s Rule approximation to \( \int_0^1 e^{x^2} \, dx \) is accurate to within 0.00001?

Solution.

\[
f(x) = e^{x^2} \implies f'(x) = 2xe^{x^2} \implies f''(x) = 2e^{x^2} + 4xe^{x^2} = 2(1 + 2x^2)e^{x^2}
\]

\[
f^{(3)}(x) = 2(4x)e^{x^2} + 4x(1 + 2x^2)e^{x^2} = 4(3x + 2x^3)e^{x^2}
\]

\[
f^{(4)}(x) = 4(3 + 6x^2)e^{x^2} + 8x(3x + 2x^3)e^{x^2} = 4(3 + 12x^2 + 4x^4)e^{x^2}
\]

On the interval \( 0 \leq x \leq 1, \) \( e^{x^2} \) is never bigger than \( e \) and \( 3 + 12x^2 + 4x^4 \) is never bigger than \( 3 + 12 + 4 = 19, \) so \( |f^{(4)}(x)| \) is never bigger than \( 4 \times 19 \times e = 76e. \) Hence error in Simpson’s rule is at most \( \frac{76e \times 10^5}{180n^4}. \) This error is smaller than \( 10^{-5} \) if

\[
\frac{76e \times 10^5}{180n^4} \leq 10^{-5} \iff n^4 \geq \frac{76e \times 10^5}{180} \iff n \geq \sqrt[4]{\frac{76e \times 10^5}{180}} = 18.4
\]

For Simpson’s rule, \( n \) must be even so take \( n \geq 20. \)

50 a) State Simpson’s Rule to approximate \( \int_a^b f(x) \, dx. \)

b) Use Simpson’s Rule to compute \( \int_0^2 x^3 \, dx \) with \( n = 4. \)

Solution. a) Let \( n \) be a strictly positive even integer and \( h = \frac{b-a}{n}. \) Define \( x_j = a + jh. \) Simpson’s Rule is

\[
\int_a^b f(x) \, dx \approx \frac{h}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]
\]

b) When \( a = 0, \) \( b = 2 \) and \( n = 4 \) we have \( h = \frac{1}{2}. \)

\[
\int_0^2 x^3 \, dx \approx \frac{1}{6} [f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + f(2)]
\]

\[
= \frac{1}{6} [0 + 4(0.5)^3 + 2(1)^3 + 4(1.5)^3 + 2^3] = 4
\]

51) Suppose that \( f(x) \) has a continuous second derivative that obeys \( |f''(x)| \leq K \) for all \( a \leq x \leq b. \) Let \( M_n \) be the result of applying the midpoint rule to \( \int_a^b f(x) \, dx \) with \( n \) subintervals. Prove that

\[
\left| \int_a^b f(x) \, dx - M_n \right| \leq \frac{K(b-a)^3}{24n^2}
\]

Hint: use the error estimate for the tangent line approximation to show that

\[
|f(x) - f(m) - f'(m)(x-m)| \leq \frac{K}{2}(x-m)^2
\]
Solution. Set, for \( j = 0, 1, 2, \ldots, n \), \( x_j = a + \frac{b-a}{n} j \). This partitions \( a \leq x \leq b \) into \( n \) equal subintervals, each of width \( \frac{b-a}{n} \). Also, let \( m_j = x_j + \frac{b-a}{2n} \) be the midpoint of the \( j \)th interval. The midpoint rule approximates \( \int_{x_j}^{x_{j+1}} f(x) \, dx \) by \( \int_{x_j}^{x_{j+1}} f(m_j) \, dx = f(m_j) \frac{b-a}{n} \).

The error in this approximation is

\[
e_j = \int_{x_j}^{x_{j+1}} f(x) \, dx - f(m_j) \frac{b-a}{n} = \int_{x_j}^{x_{j+1}} f(x) \, dx - \int_{x_j}^{x_{j+1}} f(m_j) \, dx = \int_{x_j}^{x_{j+1}} [f(x) - f(m_j)] \, dx
\]

Taylor expanding \( f(x) \) about \( m_j \) to first order (with a second order error) gives

\[
f(x) = f(m_j) + f'(m_j)(x - m_j) + \frac{1}{2} f''(z)(x - m_j)^2
\]

for some \( z \) between \( x \) and \( m_j \). By hypothesis \( |f''(z)| \leq K \) so that

\[
|f(x) - f(m_j) - f'(m_j)(x - m_j)| \leq \frac{1}{2} K (x - m_j)^2
\]

Furthermore, subbing \( y = x - m_j \),

\[
\int_{x_j}^{x_{j+1}} f'(m_j)(x - m_j) \, dx = f'(m_j) \int_{\frac{b-a}{2n}}^{\frac{b-a}{2n}} y \, dy = 0
\]

So, the error arising in the \( j \)th interval

\[
|e_j| = \left| \int_{x_j}^{x_{j+1}} [f(x) - f(m_j)] \, dx \right| = \left| \int_{x_j}^{x_{j+1}} [f(x) - f(m_j) - f'(m_j)(x - m_j)] \, dx \right| \\
\leq \int_{x_j}^{x_{j+1}} |f(x) - f(m_j) - f'(m_j)(x - m_j)| \, dx \leq \int_{x_j}^{x_{j+1}} \frac{1}{2} K (x - m_j)^2 \, dx \\
= \frac{1}{2} K \left( \frac{x-m_j}{3} \right)^3 \bigg|_{x_j}^{x_{j+1}} = 2 \times \frac{1}{2} K \left( \frac{x_{j+1}-m_j}{3} \right)^3 = \frac{1}{3} K \left( \frac{b-a}{2n} \right)^3 = \frac{1}{24} K \left( \frac{b-a}{n^3} \right)
\]

The total error, from all \( n \) intervals is

\[
\left| \int_a^b f(x) \, dx - M_n \right| \leq |e_1| + |e_2| + \cdots + |e_n| \leq n \frac{1}{24} K \left( \frac{b-a}{n^3} \right) \leq \frac{K(b-a)^3}{24n^2}
\]
52) Suppose that \( f(x) \) has a continuous second derivative that obeys \( |f''(x)| \leq K \) for all \( a \leq x \leq b \). Let \( T_n \) be the result of applying the Trapezoidal Rule to \( \int_a^b f(x) \, dx \) with \( n \) subintervals. Use the generalization of the Mean Value Theorem given below to prove that
\[
\left| \int_a^b f(x) \, dx - T_n \right| \leq \frac{K(b-a)^3}{12n^2}
\]

Solution. Set, for \( j = 0, 1, 2, \ldots, n \), \( x_j = a + \frac{b-a}{n}j \). This partitions \( a \leq x \leq b \) into \( n \) equal subintervals, each of width \( \frac{b-a}{n} \). The Trapezoidal Rule approximates \( \int_{x_j}^{x_{j+1}} f(x) \, dx \) by \( \frac{1}{2} \left[ f(x_j) + f(x_{j+1}) \right] \frac{b-a}{n} \), which is the area between \( x_j \) and \( x_{j+1} \) under the straight line through \((x_j, f(x_j))\) and \((x_{j+1}, f(x_{j+1}))\). The straight line through \((x_j, f(x_j))\) and \((x_{j+1}, f(x_{j+1}))\) is \( y = p(x) = f(x_j) + \frac{f(x_{j+1}) - f(x_j)}{x_{j+1} - x_j}(x - x_j) \). Note that
\[
\int_{x_j}^{x_{j+1}} p(x) \, dx = f(x_j)[x_{j+1} - x_j] + \frac{f(x_{j+1}) - f(x_j)}{x_{j+1} - x_j} \frac{1}{2} (x_{j+1} - x_j)^2 = \frac{1}{2} [f(x_j) + f(x_{j+1})][x_{j+1} - x_j]
\]
as desired. The error in this approximation is
\[
e_j = \int_{x_j}^{x_{j+1}} f(x) \, dx - \frac{1}{2} [f(x_j) + f(x_{j+1})] \frac{b-a}{n} = \int_{x_j}^{x_{j+1}} f(x) \, dx - \int_{x_j}^{x_{j+1}} p(x) \, dx = \int_{x_j}^{x_{j+1}} \left[ f(x) - p(x) \right] \, dx
\]
By the generalization of the Mean Value Theorem given below, with \( n = 1 \), \( y_0 = x_j \) and \( y_1 = x_{j+1} \)
\[
f(x) - p(x) = (x - x_j)(x - x_{j+1}) \frac{f''(c_x)}{2}
\]
for some \( c_x \) between \( x \) and \( m_j \). By hypothesis \( |f''(c_x)| \leq K \) so that
\[
|f(x) - p(x)| \leq \frac{1}{2} K |x - x_j||x - x_{j+1}| = \frac{1}{2} K (x - x_j)(x_{j+1} - x)
\]
for all \( x_j \leq x \leq x_{j+1} \). So, the error arising in the \( j \)th interval
\[
|e_j| = \left| \int_{x_j}^{x_{j+1}} \left[ f(x) - p(x) \right] \, dx \right| \leq \int_{x_j}^{x_{j+1}} \frac{1}{2} K (x - x_j)(x_{j+1} - x) \, dx
\]
Subbing \( t = x - x_j \), \( dt = dx \), \( x_{j+1} - x_j = \frac{b-a}{n} \),
\[
|e_j| \leq \frac{1}{2} K \int_0^{\frac{b-a}{n}} t \left( \frac{b-a}{n} - t \right) \, dt = \frac{1}{2} K \left[ \frac{b-a}{n} \frac{t^2}{2} - \frac{t^3}{3} \right]_0^{\frac{b-a}{n}} = \frac{1}{2} K \left[ \frac{(b-a)^3}{n^3} \left( \frac{1}{2} - \frac{1}{3} \right) \right] = \frac{1}{12} K \frac{(b-a)^3}{n^3}
\]
The total error, from all \( n \) intervals is
\[
\left| \int_a^b f(x) \, dx - M_n \right| \leq |e_1| + |e_2| + \cdots + |e_n| \leq n \frac{1}{12} K \frac{(b-a)^3}{n^3} \leq \frac{K(b-a)^3}{12n^2}
\]
Theorem (Mean Value Theorem for Interpolation)

Let \( a \leq y_0 < y_1 < y_2 < \cdots < y_n \leq b \). Let \( f(y) \) and its first \( n \) derivatives be continuous on \( a \leq y \leq b \) and let \( f^{(n+1)}(y) \) exist on \( a < y < b \). Suppose that \( p(y) \) is a polynomial of degree at most \( n \) that coincides with \( f(y) \) at \( y_0, \cdots, y_n \). That is, assume that

\[
f(y_0) = p(y_0), \; f(y_1) = p(y_1), \; \cdots, \; f(y_n) = p(y_n)
\]

Then, for each \( a \leq y \leq b \), there is a number \( a < c_y < b \), depending on \( y \), such that

\[
f(y) - p(y) = (y - y_0)(y - y_1)\cdots(y - y_n)\frac{f^{(n+1)}(c_y)}{(n+1)!}
\]

Proof: If \( y \) is one of \( y_0, \cdots, y_n \) then both of \( f(y) - p(y) \) and \( (y - y_0)(y - y_1)\cdots(y - y_n)\frac{f^{(n+1)}(c_y)}{(n+1)!} \) are zero, for any \( c_y \), and hence are equal. So it suffices to consider a \( y \) that is different from all of \( y_0, \cdots, y_n \). Fix any such \( y \) and define

\[
g(z) = f(z) - p(z) - \frac{(z - y_0)\cdots(z - y_n)}{(y - y_0)\cdots(y - y_n)}[f(y) - p(y)]
\]

Note that \( g(z) \) is zero for at least \( n+2 \) different values of \( z \), namely, \( z = y \) and \( z = y_0, \cdots, y_n \). The Mean Value Theorem implies that \( \frac{dg}{dz} \) has a zero between any two successive zeroes of \( g(z) \). So \( \frac{dg}{dz}(z) \) is zero for at least \( n+1 \) different values of \( z \). By the Mean Value Theorem, \( \frac{d^2g}{dz^2} \) has a zero between any two successive zeroes of \( \frac{dg}{dz} \). So \( \frac{d^2g}{dz^2}(z) \) is zero for at least \( n \) different values of \( z \). Continuing in this way, we see that \( \frac{d^{n+1}g}{dz^{n+1}} \) has at least one zero. Call it \( c_y \).

As \( p(z) \) is a polynomial in \( z \) of degree at most \( n \), \( \frac{d^{n+1}g}{dz^{n+1}} = 0 \). Recall that \( y \) is being held fixed in this computation. So \( \frac{(z-y_0)\cdots(z-y_n)}{(y-y_0)\cdots(y-y_n)}[f(y) - p(y)] \) is a polynomial in \( z \) of degree \( n+1 \) whose degree \( n+1 \) term is \( \frac{z^{n+1}}{(y-y_0)\cdots(y-y_n)}[f(y) - p(y)] \). Hence

\[
\frac{d^{n+1}}{dz^{n+1}} \frac{(z-y_0)\cdots(z-y_n)}{(y-y_0)\cdots(y-y_n)}[f(y) - p(y)] = \frac{(n+1)!}{(y-y_0)\cdots(y-y_n)}[f(y) - p(y)]
\]

and

\[
0 = \frac{d^{n+1}}{dz^{n+1}}(c_y) = f^{(n+1)}(c_y) - \frac{(n+1)!}{(y-y_0)\cdots(y-y_n)}[f(y) - p(y)]
\]

Moving \( f^{(n+1)}(c_y) \) to the other side of the equation and cross multiplying gives

\[
f(y) - p(y) = \frac{(y-y_0)\cdots(y-y_n)}{(n+1)!} f^{(n+1)}(c_y)
\]

as desired
53) Let \( I = \int_{\pi/6}^{\pi/2} \ln(\sin x) \, dx \).

a) Determine the maximum, \( K \), of \( |f''(x)| \), for \( \frac{\pi}{6} \leq x \leq \frac{\pi}{2} \), where \( f(x) = \ln(\sin x) \).

b) How large should \( n \) be in order that the approximation \( I \approx M_n \) be accurate to within \( 10^{-4} \)?

**Solution.** a) 

\[
\begin{align*}
  f(x) &= \ln(\sin x) \\
  \implies f'(x) &= \frac{\cos x}{\sin x} = \cot x \\
  \implies f''(x) &= -\csc^2 x = -\frac{1}{\sin^2 x}
\end{align*}
\]

For \( \frac{\pi}{6} \leq x \leq \frac{\pi}{2} \), \( \frac{\pi}{6} \leq \sin x \leq 1 \). So the largest value of \( |f''(x)| \) on the interval \( \frac{\pi}{6} \leq x \leq \frac{\pi}{2} \) occurs at \( x = \frac{\pi}{6} \) and is \( \frac{1}{(1/2)^2} = 4 = K \).

b) We wish to choose \( n \) so that 

\[
K(b-a)^3 \leq 10^{-4}
\]

In this case \( K = 4 \), \( a = \frac{\pi}{6} \) and \( b = \frac{\pi}{2} \) so 

\[
\frac{4(\pi/3)^3}{24n^2} \leq 10^{-4} \iff n^2 \geq \frac{4(\pi/3)^3}{24 \cdot 10^4} = \frac{\pi^3}{3^2 \cdot 2^4} \iff n \geq \frac{\pi^{3/2}}{\sqrt{2}} \approx 43.75
\]

So \( n \geq 44 \) will do the job.

54) Find the values of \( p \) for which each of the following integrals converges and evaluate the integrals for those values of \( p \).

a) \( \int_0^1 \frac{dx}{x^p} \)

b) \( \int_1^\infty \frac{dx}{x^p} \)

**Solution.** a) 

\[
\int_0^1 \frac{dx}{x^p} = \lim_{\epsilon \to 0^+} \int_{\epsilon}^{1} \frac{dx}{x^p} = \lim_{\epsilon \to 0^+} \left\{ \left[ \frac{x^{1-p}}{1-p} \right]_{\epsilon}^{1} \right\} = \lim_{\epsilon \to 0^+} \left\{ \ln x \right\}_{\epsilon}^{1} = \begin{cases} 
\frac{1}{1-p} & \text{if } p < 1 \\
\infty & \text{if } p > 1 \\
\infty & \text{if } p = 1
\end{cases}
\]

In conclusion, \( \int_0^1 \frac{dx}{x^p} = \frac{1}{1-p} \) if \( p < 1 \) and diverges otherwise.

b) 

\[
\int_1^\infty \frac{dx}{x^p} = \lim_{R \to \infty} \int_1^R \frac{dx}{x^p} = \lim_{R \to \infty} \left\{ \left[ \frac{x^{1-p}}{1-p} \right]_1^R \right\} = \lim_{R \to \infty} \left\{ \ln x \right\}_1^R = \begin{cases} 
-\frac{1}{1-p} & \text{if } p > 1 \\
\infty & \text{if } p < 1 \\
\infty & \text{if } p = 1
\end{cases}
\]
In conclusion, \( \int_1^\infty \frac{dx}{x^p} = \frac{1}{p-1} \) if \( p > 1 \) and diverges otherwise.

55) Evaluate \( \int_{-10}^{-\frac{3}{2}} \frac{1}{(2x+3)^2} \, dx \).

**Solution.** Trick question! The integrand has a singularity when \( 2x + 3 = 0 \) or \( x = -\frac{3}{2} \). The domain of integration has to be split up into two parts: \(-10 < x < -\frac{3}{2} \) and \(-\frac{3}{2} < x < 1 \).

\[
\int_{-\frac{3}{2}}^{-10} \frac{1}{(2x+3)^2} \, dx = \lim_{a \to -\frac{3}{2}^-} \frac{1}{2a+3} - \frac{1}{2} \left( \frac{1}{2a+3} \right)_{-10}
\]

\[
= \lim_{a \to -\frac{3}{2}^-} \left[ \frac{1/2}{2a+3} - \frac{1/2}{2+3} \right] = \infty
\]

since \( 2a + 3 \) approaches zero as \( a \) tends to \(-\frac{3}{2} \). So the integral diverges. Similarly,

\[
\int_{-\frac{3}{2}}^{1} \frac{1}{(2x+3)^2} \, dx = \lim_{a \to -\frac{3}{2}^-} \frac{1}{2a+3} - \frac{1}{2} \left( \frac{1}{2a+3} \right)_{-10}
\]

\[
= \lim_{a \to -\frac{3}{2}^-} \left[ \frac{1/2}{2a+3} - \frac{1/2}{2+3} \right] = \infty
\]

and the other part of the integral diverges too.

56) Determine if the following integrals converge or diverge. Justify your answers.

a) \( \int_0^\infty \frac{dx}{x^{1/3}(1+x)} \)  

b) \( \int_0^\infty \frac{dx}{x^{1/3}(1+x^{1/2})} \)

**Solution.** With both of these integrals, there are two potential problems which could cause the integrals to diverge (i.e. become infinite). First, the integrands become infinite at \( x = 0 \) and secondly \( x \) runs all the way to \( x = \infty \). To separate these two potential problems, split up the domain of integration into \( 0 \leq x \leq 1 \) and \( 1 < x \leq \infty \).

\[
\int_0^\infty \frac{dx}{x^{1/3}(1+x)} = \int_0^1 \frac{dx}{x^{1/3}(1+x)} + \int_1^\infty \frac{dx}{x^{1/3}(1+x)}
\]

\[
\int_0^\infty \frac{dx}{x^{1/3}(1+x^{1/2})} = \int_0^1 \frac{dx}{x^{1/3}(1+x^{1/2})} + \int_1^\infty \frac{dx}{x^{1/3}(1+x^{1/2})}
\]

First consider the domain of integration \( 0 \leq x \leq 1 \). On this domain \( 0 \leq \frac{1}{x^{1/3}(1+x)} \leq \frac{1}{x^{1/3}} \) and \( 0 \leq \frac{1}{x^{1/3}(1+x^{1/2})} \leq \frac{1}{x^{1/3}} \). So both integrals

\[
\int_0^1 \frac{dx}{x^{1/3}(1+x)}', \int_0^1 \frac{dx}{x^{1/3}(1+x^{1/2})} \leq \int_0^1 \frac{dx}{x^{1/3}} = \frac{2}{3} \left| \frac{1}{2} \right.
\]

\[
= \frac{3}{2}
\]
So both $\int_0^1 \frac{dx}{x^{1/3}(1+x)}$ and $\int_0^1 \frac{dx}{x^{1/3}(1+x^{1/2})}$ converge. Now for the second domain. For $x \geq 1$, $0 \leq \frac{1}{x^{1/3}(1+x^{1/2})} \leq \frac{1}{x^{1/3}}$. So

$$\int_1^\infty \frac{dx}{x^{1/3}(1+x)} \leq \int_1^\infty \frac{dx}{x^{4/3}} = \left. \frac{x^{-1/3}}{-1/3} \right|_1^\infty = 3$$

and $\int_0^\infty \frac{dx}{x^{1/3}(1+x^{1/2})}$ converges. On the other hand, for $x \geq 1$, we have $\frac{1}{x^{1/3}(1+x^{1/2})} \geq \frac{1}{2x^{5/6}}$ and hence

$$\int_1^\infty \frac{dx}{x^{1/3}(1+x^{1/2})} \geq \int_1^\infty \frac{dx}{2x^{5/6}} = \left. \frac{1}{2} \frac{x^{1/6}}{1/6} \right|_1^\infty = \infty$$

and $\int_0^\infty \frac{dx}{x^{1/3}(1+x^{1/2})}$ diverges.

57) Find the length, $L$, of the curve $y = e^x$, $0 \leq x \leq 1$.

Solution.

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = \int_0^1 \sqrt{1 + e^{2x}} \, dx$$

Make the change of variables $u = \sqrt{1 + e^{2x}}$, $du = \frac{e^{2x}}{\sqrt{1 + e^{2x}}} \, dx = \frac{u^2 - 1}{u} \, dx$.

$$L = \int_{\sqrt{2}}^{\sqrt{2}} \frac{u^2}{u^2 - 1} \, du = \int_{\sqrt{2}}^{\sqrt{2}} \left[ 1 + \frac{1/2}{u - 1} - \frac{1/2}{u + 1} \right] \, du$$

$$= \left[ u + \frac{1}{2} \ln |u - 1| - \frac{1}{2} \ln |u + 1| \right]_{\sqrt{2}}^{\sqrt{2}}$$

$$= \sqrt{1 + e^2} - \sqrt{2} + \frac{1}{2} \ln \frac{\sqrt{1 + e^2 - 1}}{\sqrt{1 + e^2 + 1}} - \frac{1}{2} \ln \frac{\sqrt{2 - 1}}{\sqrt{2 + 1}}$$

58) Find the length of the curve $12xy = 4y^4 + 3$ from $(\frac{7}{12}, 1)$ to $(\frac{67}{24}, 2)$.

Solution. On this curve $x = \frac{1}{3}y^3 + \frac{1}{4y}$. So

$$\frac{dx}{dy} = y^2 - \frac{1}{4y^2}$$

$$1 + \left(\frac{dx}{dy}\right)^2 = 1 + y^4 - \frac{1}{2} + \frac{1}{16y^4} = y^4 + \frac{1}{2} + \frac{1}{16y^4} = (y^2 + \frac{1}{4y^2})^2$$

Hence

$$\text{length} = \int_1^2 \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy = \int_1^2 \left( y^2 + \frac{1}{4y^2} \right) \, dy = \left[ \frac{y^3}{3} - \frac{1}{4y} \right]_1^2 = \frac{29}{24}$$
59) Find the length of the curve \( x^{2/3} + y^{2/3} = 1 \).

**Solution.** This equation does not change if you substitute \( x \to -x \) or \( y \to -y \). So the curve is invariant under reflection in the \( x- \) or \( y- \)axes. It suffices to find the length of the part of the curve in the first quadrant and multiply by 4.

![Graph of the curve](image)

On this curve \( y = (1 - x^{2/3})^{3/2} \). So

\[
\frac{dy}{dx} = \frac{3}{2} (1 - x^{2/3})^{1/2} (- \frac{2}{3} x^{-1/3}) = -(1 - x^{2/3})^{1/2} x^{-1/3}
\]

\[ 1 + (\frac{dy}{dx})^2 = 1 + (1 - x^{2/3}) x^{-2/3} = x^{-2/3} \]

Hence

\[
\text{length} = 4 \int_0^1 \sqrt{1 + (\frac{dy}{dx})^2} \, dx = 4 \int_0^1 x^{-1/3} \, dx = 4 \frac{3}{2} x^{2/3} \bigg|_0^1 = 6
\]

60) Set up two integrals, one of which is improper, representing the part of the curve \( y^3 = x^2 \) between \((0, 0)\) and \((1, 1)\). Evaluate both integrals.

**Solution.** Since \( 3y^2 \, dy = 2x \, dx \)

\[
\frac{dy}{dx} = \frac{2x}{3y^2} = \frac{2}{3x^{1/3}}, \quad \frac{dx}{dy} = \frac{3y^2}{2x} = \frac{3}{2} y^{1/2}
\]

\[
L = \int_0^1 \sqrt{1 + (\frac{dy}{dx})^2} \, dx = \int_0^1 \sqrt{1 + (\frac{dx}{dy})^2} \, dy
\]

\[
= \int_0^1 \sqrt{1 + \frac{4}{9x^{2/3}}} \, dx = \int_0^1 \sqrt{1 + \frac{9}{4} y} \, dy
\]

To evaluate the integral over \( y \), substitute \( z = 1 + \frac{9}{4} y \):

\[
L = \int_1^{13/4} \sqrt{z} \, \frac{4}{9} \, dz = \frac{4}{9} \left[ \frac{2}{3} z^{3/2} \right]_1^{13/4} = \frac{8}{27} \left[ \frac{13^{3/2}}{8} - 1 \right]
\]

To evaluate the integral over \( x \), which is improper because of the singularity at \( x = 0 \), substitute \( x = y^{3/2} \), \( dx = \frac{3}{2} y^{1/2} \, dy \). This yields the \( y \)-integral that we just evaluated.

61) Find the area of the surface obtained by rotating the given curve about the \( x \)-axis.

a) \( y = \frac{1}{4} x^2 - \frac{1}{2} \ln x \), \( 1 \leq x \leq 4 \)  

b) \( 8y^2 = x^2 (1 - x^2) \), \( 0 \leq x \leq 1 \)
Solution. a)

\[
\frac{dy}{dx} = \frac{1}{2} x - \frac{1}{2x}
\]

\[
 \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}(x^2 - 2 + \frac{1}{x^2})
\]

\[
1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{4}(x^2 - 2 + \frac{1}{x}) = \frac{1}{4}(x^2 + 2 + \frac{1}{x^2}) = \frac{1}{4}(x + \frac{1}{x})^2
\]

So

\[
\text{Area} = \int_1^4 2\pi y(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = 2\pi \int_1^4 \left(\frac{1}{4} x^2 - \frac{1}{2} \ln x\right) \frac{1}{2} \left(x + \frac{1}{x}\right) \, dx
\]

\[
= \frac{\pi}{2} \int_1^4 \left(\frac{1}{2} x^3 + \frac{1}{2} x - x \ln x - \frac{1}{x} \ln x\right) \, dx
\]

\[
= \frac{\pi}{2} \left[\frac{x^4}{8} + \frac{x^2}{2} - \frac{x^2}{2} \ln x + \frac{x^2}{4} - \frac{1}{2} (\ln x)^2\right]_1^4
\]

\[
= \pi \left[\frac{315}{16} - 8\ln 2 - (\ln 2)^2\right]
\]

b) Differentiating both sides of \(8y(x)^2 = x^2(1 - x^2)\) with respect to \(x\),

\[
16y \frac{dy}{dx} = 2x - 4x^3 = 2x(1 - 2x^2) \quad \implies \quad \frac{dy}{dx} = \frac{x}{8y}(1 - 2x^2)
\]

\[
\left(\frac{dy}{dx}\right)^2 = \frac{x^2}{64y^2}(1 - 2x^2)^2 = \left(\frac{1 - 2x^2}{8}\right)^2
\]

\[
1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{(1 - 2x^2)^2}{8} = \frac{8 - 8x^2 + 1 - 4x^2 + 4x^4}{8(1 - x^2)} = \frac{9 - 12x^2 + 4x^4}{8(1 - x^2)} = \frac{(3 - 2x^2)^2}{8(1 - x^2)}
\]

So

\[
\text{Area} = \int_0^1 2\pi y(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx = 2\pi \int_0^1 \frac{x\sqrt{1 - x^2}}{\sqrt{8}} \frac{3 - 2x^2}{\sqrt{8(1 - x^2)}} \, dx
\]

\[
= \frac{\pi}{4} \int_0^1 (3x - 2x^3) \, dx = \frac{\pi}{4} \left[\frac{3}{2} x^2 - \frac{1}{2} x^4\right]_0^1
\]

\[
= \frac{\pi}{4}
\]

62) Let \(L\) be the length of the part of the curve \(y = e^{-x}\) lying between \(x = 0\) and \(x = 4\). Use the Trapezoidal Rule with four subintervals to estimate \(L\).

Solution. Let \(f(x) = e^{-x}\). Then \(f'(x) = -e^{-x}\) so \(L = \int_0^4 \sqrt{1 + f'(x)^2} \, dx = \int_0^4 \sqrt{1 + e^{-2x}} \, dx\). Denoting \(g(x) = \sqrt{1 + e^{-2x}}\), the Trapezoidal Rule with \(\Delta x = 1\) gives

\[
L \approx \Delta x \left[rac{1}{2} g(0) + g(1) + g(2) + g(3) + \frac{1}{2} g(4)\right] \approx .707 + 1.066 + 1.009 + 1.001 + 0.500 = 4.283
\]

63) At what point does the curve \(x = 1 - 2 \cos^2 t, \ y = (\tan t)(1 - 2 \cos^2 t)\) cross itself? Find the equations of both tangents at that point.
Solution. Suppose that the curve is at the same point at time $t_1$ as it is at time $t_2$. Then

$$x(t_1) = x(t_2) \iff 1 - 2 \cos^2 t_1 = 1 - 2 \cos^2 t_2 \iff \cos^2 t_1 = \cos^2 t_2 \iff \cos t_1 = \pm \cos t_2$$

and

$$y(t_1) = y(t_2) \implies (\tan t_1)(1 - 2 \cos^2 t_1) = (\tan t_2)(1 - 2 \cos^2 t_2)$$

This is the case if $1 - 2 \cos^2 t_1 = 0$ or $\cos t_1 = \frac{1}{\sqrt{2}}$, since then $1 - 2 \cos^2 t_2 = 0$ automatically and both $y(t_1)$ and $y(t_2)$ are zero. For example $t_1 = \frac{\pi}{4}$ and $t_2 = \frac{3\pi}{4}$ does the job. At both these times the curve is at $(0,0)$. Since

$$x'(t) = 4 \sin t \cos t \quad y'(t) = (\sec^2 t)(1 - 2 \cos^2 t) + (\tan t)(4 \sin t \cos t)$$

$$\implies \frac{dy}{dx} = \frac{(\sec^2 t)(1 - 2 \cos^2 t)}{4 \sin t \cos t} + \tan t$$

When $t = \frac{\pi}{4}$ or $\frac{3\pi}{4}$, the term $\frac{(\sec^2 t)(1 - 2 \cos^2 t)}{4 \sin t \cos t} = 0$. Hence

$$\frac{dy}{dx} \bigg|_{t=\pi/4} = \tan \frac{\pi}{4} = 1 \quad \frac{dy}{dx} \bigg|_{t=3\pi/4} = \tan \frac{3\pi}{4} = -1$$

The line through $(0,0)$ with slope $+1$ is $y = x$ and the line through $(0,0)$ with slope $-1$ is $y = -x$.

64) A string is wound around a circle of radius $r$ and then unwound while being held taut. Find the equation of the curve traced by the point $P$ at the end of the string if the initial position of $P$ is $(r,0)$.

Solution.

Use the angle $\theta$ in the figure above as the parameter. The point $T$ has coordinates $(r \cos \theta, r \sin \theta)$. The length of the line $TP$ is the same as the length of the arc of the circle from $(r,0)$ to $T$. This arc is the fraction $\frac{\theta}{2\pi}$ of the full circle, which has circumference $2\pi r$. So the line $TP$ has length $\frac{\theta}{2\pi}2\pi r = \theta r$. The line $OT$ is perpendicular to the line $TP$ and the line $TS$ is parallel to the $x$–axis. So

$$\theta + \frac{\pi}{2} + \alpha = \pi \implies \alpha = \frac{\pi}{2} - \theta$$
Consequently, the \( x \)-coordinate of \( P \) is the \( x \)-coordinate of \( T \), which is \( r \cos \theta \) plus the length of \( TS \) which is \( |TP| \cos \alpha = r\theta \sin \theta \). The \( y \)-coordinate of \( P \) is the \( y \)-coordinate of \( T \), which is \( r \sin \theta \) minus the length of \( SP \) which is \( |TP| \sin \alpha = r\theta \cos \theta \). So

\[
\begin{align*}
x &= r( \cos \theta + \theta \sin \theta ) \\
y &= r( \sin \theta - \theta \cos \theta )
\end{align*}
\]

65) A cow is tied to a silo with radius \( r \) by a rope that is just long enough to reach the opposite side of the silo. Find the area available for grazing by the cow.

**Solution.** Imagine the cow walking with the rope pulled taut, starting from the opposite side of the silo. At first, the cow follows the parametrized path that was found in Problem #64 above. This happens until the parameter value \( \theta \) reaches \( \pi \). At his point, the rope is a vertical straight line segment that is tangent to the silo (as illustrated in the figure above). From this point the cow just swings in a semicircle until the rope is pointing straight down. The length of the rope is \( \pi r \), half the circumference of the silo. So the top half of the grazing area is (a) the area between \( \{ r( \cos \theta + \theta \sin \theta , \sin \theta - \theta \cos \theta ) \mid 0 \leq \theta \leq \pi \} \) and the line \( x = -r \), plus (b) the area of one quarter of a circle of radius \( \pi r \) minus (c) the area of one half of a circle of radius \( r \) (since the cow cannot graze inside of the silo itself). The thin horizontal strip shown in the figure above has length \( x + r = r( \cos \theta + \theta \sin \theta + 1 ) \) and width \( dy = \frac{dy}{d\theta} d\theta \). So the the area between \( \{ r( \cos \theta + \theta \sin \theta , \sin \theta - \theta \cos \theta ) \mid 0 \leq \theta \leq \pi \} \) and the line \( x = -r \) is

\[
\begin{align*}
\int_0^\pi \left[ x(\theta) + r \right] \frac{dy}{d\theta} d\theta &= \int_0^\pi r^2 \left[ \cos \theta + \theta \sin \theta + 1 \right] \left( \cos \theta - \cos \theta + \theta \sin \theta \right) d\theta \\
&= r^2 \int_0^\pi \left( \theta \sin \theta + \theta \sin \theta \cos \theta + \theta^2 \sin^2 \theta \right) d\theta \\
&= \frac{1}{2} r^2 \int_0^\pi \left( 2\theta \sin \theta + \theta \sin(2\theta) - \theta^2 \cos(2\theta) + \theta^2 \right) d\theta \\
&= \frac{1}{2} r^2 \left[ -\frac{1}{2} \theta^2 \sin(2\theta) - \theta \cos(2\theta) + \frac{1}{2} \sin(2\theta) - 2\theta \cos \theta + 2 \sin \theta + \frac{\theta^3}{3} \right]_0^\pi \\
&= \frac{1}{2} r^2 \left[ -\pi + 2\pi + \frac{\pi^3}{3} \right] = \frac{1}{2} r^2 \left[ \pi + \frac{\pi^3}{3} \right]
\end{align*}
\]
So the area of the top half is
\[
\frac{1}{2} r^2 \left[ \pi + \frac{\pi^3}{3} \right] + \frac{1}{4} \pi (\pi r)^2 - \frac{1}{2} \pi r^2 = \frac{5}{12} \pi^3 r^2
\]

The full area is twice this, or \( \frac{5}{6} \pi^3 r^2 \).

66) At 2:00 a.m. the animal, \textit{studentia enubriatus}, having contracted the disease, \textit{munchia}, leaves its dwelling to forage for food. At time \( t \), its \( x \)-coordinate (in m) is \( 60e^t \sin 2\pi t \) and its \( y \)-coordinate (in m) is \( 60e^t \cos 2\pi t \). (The time \( t \) is in hours and \( t = 0 \) corresponds to 2:00 a.m.) If it still has not found any food at 4:00 a.m.

a) sketch the path that it has followed and

b) determine how far has it traveled.

\textbf{Solution.} a)

\[
\begin{align*}
\frac{dx}{dt} & = 60 e^t \sin 2\pi t + 120 \pi e^t \cos 2\pi t \\
\frac{dy}{dt} & = 60 e^t \cos 2\pi t - 120 \pi e^t \sin 2\pi t \\
\Rightarrow \left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2 & = (60 e^t)^2 \left[ \sin^2 2\pi t + 4\pi \sin 2\pi t \cos 2\pi t + 4\pi^2 \cos^2 2\pi t \\
& \quad + \cos^2 2\pi t - 4\pi \sin 2\pi t \cos 2\pi t + 4\pi^2 \sin^2 2\pi t \right] \\
& = (60 e^t)^2 \left[ 1 + 4\pi^2 \right]
\end{align*}
\]

Hence
\[
L = \int_{0}^{2} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} \, dt = 60 \sqrt{1 + 4\pi^2} \int_{0}^{2} e^t \, dt = 60 \sqrt{1 + 4\pi^2} \left[ e^2 - 1 \right]
\]

67) Find the length of the parametric curve \( x = \frac{3}{4} t^4 \), \( y(t) = \frac{1}{6} t^6 \), \( t \geq 0 \) from the point (0,0) to the point \( \left( 12, \frac{32}{3} \right) \).
Solution. The points $(0,0)$ and $(12,\frac{32}{3})$ correspond to $t = 0$ and $t = 2$ respectively. So, the length is

\[
\int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt = \int_0^2 \sqrt{9t^6 + t^{10}} \, dt = \int_0^2 t^3 \sqrt{9 + t^4} \, dt
\]

Make the change of variables $u = t^4 + 9$, $du = 4t^3 \, dt$. When $t = 0$, $u = 9$ and when $t = 2$, $u = 25$, so the length is

\[
\int_9^{25} \sqrt{u} \, \frac{du}{4} = \frac{1}{4} \frac{u^{3/2}}{3/2} \bigg|_9^{25} = \frac{1}{6} [5^3 - 3^3] = \frac{49}{3}
\]

68) A curve $C$ has parametric representation $x = 6t + 1$, $y = t^3 - 2t$.

a) Find all points on $C$ at which the tangent line is perpendicular to $3x + 5y - 8 = 0$.

b) Does the curve have a vertical tangent at any point?

c) Find all local minima and maxima for $y$.

Solution. a) $3x + 5y - 8 = 0$, or equivalently, $y = \frac{8}{5} - \frac{3}{5}x$, is a straight line with slope $-\frac{3}{5}$. To be perpendicular to it, another line must have slope $\frac{5}{3}$. The slope of the curve is $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3t^2 - 2}{6}$. This slope is $\frac{5}{3}$ when

\[
\frac{3t^2 - 2}{6} = \frac{5}{3} \iff 3t^2 - 2 = 10 \iff t^2 = 4 \iff t = \pm 2
\]

The points on $C$ have $t = \pm 2$ are $(-11, -4)$ and $(13, 4)$.

b) The curve $C$ has vertical tangent when $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3t^2 - 2}{6}$ is infinite. This never happens.

c) Observe that $\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3t^2 - 2}{6}$ is zero for $t = \pm \sqrt{2/3}$, positive for $t < -\sqrt{2/3}$ and $t > \sqrt{2/3}$ and negative for $-\sqrt{2/3} < t < -\sqrt{2/3}$. That is, $y(t)$ increases for $-\infty < t < -\sqrt{2/3}$ then decreases for $-\sqrt{2/3} < t < \sqrt{2/3}$ and then increases for $t > \sqrt{2/3}$. So $y$ has a local maximum when $t = -\sqrt{2/3}$ (so that $y = \frac{4\sqrt{6}}{9}$) and a local minimum when $t = \sqrt{2/3}$ (so that $y = -\frac{4\sqrt{6}}{9}$).

69 a) Sketch $r = 3 + 2\sin \theta$.

b) Describe $r = 2(\sin \theta + \cos \theta)$.

Solution. a) As

- $\theta$ runs from 0 to $\frac{\pi}{2}$, $\sin \theta$ runs from 0 to 1, so $r$ runs from 3 to 5.
- $\theta$ runs from $\frac{\pi}{2}$ to $\pi$, $\sin \theta$ runs from 1 to 0, so $r$ runs from 5 to 3.
- $\theta$ runs from $\pi$ to $\frac{3\pi}{2}$, $\sin \theta$ runs from 0 to $-1$, so $r$ runs from 3 to 1.
- $\theta$ runs from $\frac{3\pi}{2}$ to $2\pi$, $\sin \theta$ runs from $-1$ to 0, so $r$ runs from 1 to 3.

This gives the figure
b)  
\[ r = 2(\sin \theta + \cos \theta) \quad \implies \quad r^2 = 2(r \sin \theta + r \cos \theta) \quad \implies \quad x^2 + y^2 = 2(y + x) \]
\[ \implies \quad (x - 1)^2 + (y - 1)^2 = 2 \]

This is a circle of radius \( \sqrt{2} \) centered on \((1, 1)\).

70) Find the area of the region formed by the curve parametrized by the equations 
\[ x = 4a \cos t, \quad y = a\sqrt{2} \sin t \cos t \]
for \(-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}\).

**Solution.**

Note that as \( t \) runs from \(-\frac{\pi}{2} \) to 0, \( x \) runs from 0 to \( 4a \) and \( y = \frac{a}{\sqrt{2}} \sin(2t) \) runs from 0 to \(-\frac{a}{\sqrt{2}} \) (at \( t = -\frac{\pi}{4} \)) and then back to 0. As \( t \) runs from 0 to \( \frac{\pi}{2} \), \( x \) runs from \( 4a \) back to 0 and \( y \) runs from 0 to \( \frac{a}{\sqrt{2}} \) (at \( t = \frac{\pi}{4} \)) and then back to 0. This curve is symmetric about the \( x \) axis, because \( x \) is even under \( t \to -t \) while \( y \) is odd under \( t \to -t \). So the area is twice the area above the \( x \) axis. The latter area is obtained by having \( t \) run from \( \frac{\pi}{2} \) to 0. The total area is

\[
2 \int_0^{4a} y \, dx = 2 \int_{\pi/2}^0 y(t)x'(t) \, dt = 8a^2\sqrt{2} \int_{\pi/2}^0 \sin t \cos t (-\sin t) \, dt
\]

Making the substitution \( u = \sin t \), \( du = \cos t \, dt \)

\[ \text{Area} = -8a^2\sqrt{2} \int_1^0 u^2 \, du = -8a^2\sqrt{2} \cdot \frac{u^3}{3} \bigg|_1^0 = -\frac{8\sqrt{2}}{3} a^2 \]

71) Find the area of the region lying inside both \( r = 1 \) and \( r^2 = 2 \sin 2\theta \).

**Solution.**

The two curves intersect when \( r^2 = 1 = 2 \sin 2\theta \). At these points \( \sin 2\theta = \frac{1}{2} \), so \( 2\theta = \frac{\pi}{6} \), \( \frac{5\pi}{6} \), \( \cdots \) or \( \theta = \frac{\pi}{12} \), \( \frac{5\pi}{12} \), \( \cdots \). By symmetry, we may find the area inside both \( r = 1 \) and \( r = \sqrt{2} \sin 2\theta \) with \( 0 \leq \theta \leq \frac{\pi}{4} \) and then multiply by 4. For \( 0 \leq \theta \leq \frac{\pi}{12} \), \( 2 \sin 2\theta \) is smaller than 1 so the outer boundary of the region is \( r = \sqrt{2} \sin 2\theta \). For \( \frac{\pi}{12} \leq \theta \leq \frac{\pi}{4} \),
2\sin 2\theta \text{ is larger than 1 so the outer boundary of the region is } r = 1.

The area of the region is

\[ A = 4\left[\frac{1}{2} \int_{0}^{\pi/12} \left(\sqrt{2\sin 2\theta}\right)^2 d\theta + \frac{1}{2} \int_{\pi/12}^{\pi/4} 1^2 d\theta \right] = 4 \left[ \int_{0}^{\pi/12} \sin(2\theta) \, d\theta + \frac{1}{3} \pi \right] \]

\[ = 4 \left[ -\cos \frac{2\theta}{2} \bigg|_{0}^{\pi/12} + \frac{\pi}{12} \right] = 4 \left[ \frac{1}{2} - \frac{\sqrt{3}/2}{2} + \frac{\pi}{12} \right] = 2 - \sqrt{3} + \frac{\pi}{3} \]

72) Consider the cardioid \( r = 1 - \cos \theta \) and the unit circle \( r = 1 \).

a) Graph both curves on the same graph.

b) Find the area which lies inside the cardioid, but outside the circle.

**Solution.**

The two curves intersect when \( r = 1 = 1 - \cos \theta \) or \( r = 1, \theta = \pm \frac{\pi}{2} \) or \( x = 0, y = \pm 1 \). We are to find the area of the region with \( \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2} \), \( 1 \leq r \leq 1 - \cos \theta \). This region has area

\[ \frac{1}{2} \int_{\pi/2}^{3\pi/2} \left[ (1 - \cos \theta)^2 - 1^2 \right] d\theta = \frac{1}{2} \int_{\pi/2}^{3\pi/2} \left[ \cos^2 \theta - 2 \cos \theta \right] d\theta \]

By the trig identity \( \cos^2 \theta = \frac{1+\cos(2\theta)}{2} \), the area is

\[ \frac{1}{2} \int_{\pi/2}^{3\pi/2} \left[ \frac{1}{2} + \frac{1}{2} \cos(2\theta) - 2 \cos \theta \right] d\theta = \left[ \frac{1}{4} + \frac{1}{4} \sin(2\theta) - \sin \theta \right]_{\pi/2}^{3\pi/2} = \frac{3}{2} \]

73) Sketch and find the area of the plane region that lies inside both the circle with polar equation \( r = \sin \theta \) and the circle with polar equation \( r = \sqrt{3}\cos \theta \).

**Solution.**

The two circles intersect when \( r = 0 \) and when \( r = \sin \theta = \sqrt{3}\cos \theta \). At the latter point \( \tan \theta = \sqrt{3}, \text{ so } \theta = \frac{\pi}{3} \). For \( 0 \leq \theta \leq \frac{\pi}{3} \), the outer boundary of the region is \( r = \sin \theta \). For \( \frac{\pi}{3} \leq \theta \leq \frac{\pi}{2} \), the outer boundary of the region is \( r = \sqrt{3}\cos \theta \). The area of the region is

\[ A = \frac{1}{2} \int_{0}^{\pi/3} \sin^2 \theta \, d\theta + \frac{1}{2} \int_{\pi/3}^{\pi/2} 3\cos^2 \theta \, d\theta \]

Subbing in \( \sin^2 \theta = \frac{1-\cos 2\theta}{2} \) and \( \cos^2 \theta = \frac{1+\cos 2\theta}{2} \),

\[ A = \frac{1}{4} \int_{0}^{\pi/3} (1 - \cos 2\theta) \, d\theta + \frac{3}{4} \int_{\pi/3}^{\pi/2} (1 + \cos 2\theta) \, d\theta \]

\[ = \frac{1}{4} \left[ \theta - \frac{\sin 2\theta}{2} \right]_{0}^{\pi/3} + \frac{3}{4} \left[ \theta + \frac{\sin 2\theta}{2} \right]_{\pi/3}^{\pi/2} \]

\[ = \frac{\pi}{4} - \frac{1}{8} \sin \frac{2\pi}{3} + \frac{3\pi}{4} - \frac{3\pi}{8} \sin \frac{2\pi}{3} \]

\[ = \frac{5\pi}{24} - \frac{\sqrt{3}}{4} \]

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74) Find the area of the region lying inside both \( r = 1 \) and \( r = 2 \sin 2\theta \).

**Solution.**

By symmetry, we may find the area inside both \( r = 1 \) and \( r = 2 \sin 2\theta \) with \( 0 \leq \theta \leq \frac{\pi}{4} \) and then multiply by 8. The two curves intersect when \( r = 1 = 2 \sin 2\theta \). At these points \( \sin 2\theta = \frac{1}{2} \), so \( 2\theta = \frac{\pi}{6}, \frac{5\pi}{6} \), \ldots or \( \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \ldots \). For \( 0 \leq \theta \leq \frac{\pi}{12} \), \( 2 \sin 2\theta \) is smaller than 1 so the outer boundary of the region is \( r = 2 \sin 2\theta \). For \( \frac{\pi}{12} \leq \theta \leq \frac{\pi}{4} \), \( 2 \sin 2\theta \) is larger than 1 so the outer boundary of the region is \( r = 1 \). The area of the region is

\[
A = 8 \left[ \frac{1}{2} \int_0^{\pi/12} 4 \sin^2 2\theta \, d\theta + \frac{1}{2} \int_{\pi/12}^{\pi/4} 1^2 \, d\theta \right] = 8 \left[ \int_0^{\pi/12} (1 - \cos 4\theta) \, d\theta + \frac{\pi}{2} \right]
\]

75) Evaluate \( \int \tan^{-1}(x^2) \, dx \) as a series.

**Solution.**

\[
\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n
\]

\[
\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad \text{(replaced } x \text{ by } x^2)\]

\[\tan^{-1} x = \int \frac{dx}{1+x^2} = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}\]

Because \( \tan^{-1}(0) = 0 \) the constant \( C \) must be zero and

\[
\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}
\]

\[
\tan^{-1}(x^2) = x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \frac{x^{14}}{7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1} \quad \text{(replace } x \text{ by } x^2)\]

\[
\int \tan^{-1}(x^2) \, dx = C + \frac{x^3}{3} - \frac{x^7}{3 \times 7} + \frac{x^{11}}{5 \times 11} - \frac{x^{15}}{7 \times 15} + \cdots = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n+1)(4n+3)}
\]

76) Find the sum of \( \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{6^{2n}(2n)!} \).
Solution.

\[ \cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n)!} \]

\[ \sqrt{\frac{3}{2}} = \cos \left( \frac{\pi}{6} \right) = 1 - \frac{\pi^2}{2!6^2} + \frac{\pi^4}{4!6^4} - \frac{\pi^6}{6!6^6} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!6^{2n}} \]

77) Show that \( e^x > 1 + x \) for all \( x > 0 \).

Solution.

\[ e^x = 1 + x + \frac{1}{2} x^2 + \frac{1}{3!} x^3 + \cdots + \frac{1}{n!} x^n + \cdots \]

When \( x > 0, \frac{1}{n!} x^n > 0 \) for every \( n \). Hence

\[ A = \frac{1}{2} x^2 + \frac{1}{3!} x^3 + \cdots + \frac{1}{n!} x^n + \cdots > 0 \implies e^x = 1 + x + A > 1 + x \]

78) Use series to evaluate the limit

\[ \lim_{x \to 0} \frac{1 - \cos x}{1 + x - e^x} \]

Solution.

\[ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \]

\[ 1 - \cos x = \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \cdots \]

\[ = x^2 \left( \frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - \cdots \right) \]

\[ e^x = 1 + x + \frac{1}{2} x^2 + \frac{1}{3!} x^3 + \cdots \]

\[ 1 + x - e^x = -\frac{1}{2} x^2 - \frac{1}{3!} x^3 - \cdots \]

\[ = -x^2 \left( \frac{1}{2} - \frac{1}{3!} x - \cdots \right) \]

So

\[ \lim_{x \to 0} \frac{1 - \cos x}{1 + x - e^x} = \lim_{x \to 0} \frac{x^2 \left( \frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - \cdots \right)}{-x^2 \left( \frac{1}{2} - \frac{1}{3!} x - \cdots \right)} = -\lim_{x \to 0} \frac{\frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - \cdots}{\frac{1}{2} - \frac{1}{3!} x - \cdots} = -\frac{1}{2} = \boxed{-1} \]

79) Let \( I(x) = \int_0^x t^3 e^{-t^2} dt \).

a) Find a Maclaurin series for \( I(x) \).

b) Use this series to approximate \( I(1) \), with an error less than 0.01.
Solution. a)

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]

\[ e^{-t^2} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{n!} \quad \text{(subbed in } x = -t^2) \]

\[ t^3 e^{-t^2} = t^3 - t^5 + \frac{t^7}{2!} - \frac{t^9}{3!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+3}}{n!} \]

\[ I(x) = \int_0^x \frac{\sin t}{t} \, dt = \frac{x^4}{4} - \frac{x^6}{6} + \frac{x^8}{24} - \frac{x^{10}}{120} + \frac{x^{12}}{720} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+4}}{n!(2n+4)} \]

b)

\[ I(1) = \frac{1}{4} - \frac{1}{6} + \frac{1}{218} - \frac{1}{3110} + \cdots = 0.25 - 0.1\dot{6} + 0.0625 - 0.01\dot{6} + 0.000145 - \cdots = 0.129 \]

The series for \( I(1) \) is an alternating series with decreasing successive terms. So approximating \( I(1) \) by \( \frac{1}{4} - \frac{1}{6} + \frac{1}{218} - \frac{1}{3110} \) introduces an error between 0 and \( \frac{1}{4112} = 0.000145 \).

80) Let \( I(x) = \int_0^x \frac{\sin t}{t} \, dt \).

a) Find a Maclaurin series for \( I(x) \).

b) Use this series to calculate an approximate value for \( I(0.5) \), with an error less than \( 10^{-6} \).

Solution. a)

\[ \sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} \]

\[ \frac{\sin t}{t} = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(2n+1)!} \]

\[ I(x) = \int_0^x \frac{\sin t}{t} \, dt = x - \frac{x^3}{3!3} + \frac{x^5}{5!5} - \frac{x^7}{7!7} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!(2n+1)} \]

b)

\[ I(0.5) = \frac{1}{2} - \frac{1}{2^33} + \frac{1}{2^555} - \frac{1}{2^777} + \cdots = 0.5 - 0.0069444 + 0.0000521 - 0.0000002 - \cdots = 0.4931076 \]

The series for \( I(0.5) \) is an alternating series with decreasing successive terms. So approximating \( I(0.5) \) by \( \frac{1}{2} - \frac{1}{2^33} + \frac{1}{2^555} \) introduces an error between 0 and \( -\frac{1}{2^777} = -2 \times 10^{-7} \).
81) Use a suitable series expansion to calculate

\[
\int_0^{1/4} \frac{1}{\sqrt{t(1+t^2)}} \, dt
\]

to within \(10^{-4}\).

**Solution.** Define \(I(x) = \int_0^x \frac{1}{\sqrt{t(1+t^2)}} \, dt\). Then

\[
\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n
\]

\[
\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots = \sum_{n=0}^{\infty} (-1)^n t^{2n} \quad \text{(subbed in } x = -t^2) \]

\[
\frac{1}{\sqrt{t(1+t^2)}} = \frac{1}{\sqrt{t}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{7}} - t^{1/2} + \cdots = \sum_{n=0}^{\infty} (-1)^n t^{2n-0.5}
\]

\[
I(x) = \frac{x^{1/2}}{1/2} - \frac{x^{3/2}}{3/2} + \frac{x^{5/2}}{5/2} - \frac{x^{7/2}}{7/2} + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n+0.5} \frac{2n+0.5}{2n+0.5}
\]

In particular, since \((\frac{1}{4})^{1/2} = 2^{-1}\),

\[
I(\frac{1}{4}) = \frac{2^{-1}}{1/2} - \frac{2^{-5}}{5/2} + \frac{2^{-9}}{9/2} - \frac{2^{-13}}{13/2} + \cdots
\]

\[
= 1 - \frac{2^{-4}}{5} + \frac{2^{-8}}{9} - \frac{2^{-12}}{13} + \cdots
\]

\[
= 1 - 0.0125 + 0.000434 - 0.000019 \cdots = 0.9879
\]

The series for \(I(\frac{1}{4})\) is an alternating series with decreasing successive terms. So approximating \(I(\frac{1}{4})\) by \(1 - \frac{2^{-4}}{5} + \frac{2^{-8}}{9}\) introduces an error between 0 and \(-\frac{2^{-12}}{13} = -0.000019\).

82) Find the Maclaurin series for \(\ln(1-x)\) and use it to calculate \(\ln(0.9)\), with an error less than \(10^{-5}\). Justify that your error is less than \(10^{-5}\).

**Solution.** Let \(f(x) = \ln(1-x)\). Then

\[
\frac{d}{dx} \ln(1-x) = -\frac{1}{1-x} \quad \frac{d^2}{dx^2} \ln(1-x) = -\frac{1}{(1-x)^2} \quad \frac{d^3}{dx^3} \ln(1-x) = -\frac{2}{(1-x)^3}
\]

\[
\frac{d^4}{dx^4} \ln(1-x) = -\frac{2 \times 3}{(1-x)^4} \quad \frac{d^5}{dx^5} \ln(1-x) = -\frac{2 \times 3 \times 4}{(1-x)^5} \quad \frac{d^n}{dx^n} \ln(1-x) = -\frac{(n-1)!}{(1-x)^n}
\]

In particular

\[
\ln(1-x) \bigg|_{x=0} = 0 \quad \frac{d}{dx} \ln(1-x) \bigg|_{x=0} = -1 \quad \frac{d^2}{dx^2} \ln(1-x) \bigg|_{x=0} = -1 \quad \frac{d^3}{dx^3} \ln(1-x) \bigg|_{x=0} = -1
\]

\[
\frac{d^4}{dx^4} \ln(1-x) \bigg|_{x=0} = -2 \quad \frac{d^5}{dx^5} \ln(1-x) \bigg|_{x=0} = -3! \quad \frac{d^n}{dx^n} \ln(1-x) \bigg|_{x=0} = -(n-1)!
\]

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Hence
\[ \ln(1 - x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots - \frac{1}{n}x^n + R_n(x) \]

where
\[ R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1} = -\frac{1}{n+1} \frac{1}{(1-z)^{n+1}}x^{n+1} \]

for some \( z \) between 0 and \( x \). In particular, with \( x = 0.1 \),
\[ \ln(0.9) = -\frac{1}{10} - \frac{1}{2} \frac{1}{10^2} - \frac{1}{3} \frac{1}{10^3} - \cdots - \frac{1}{n} \frac{1}{10^n} + R_n(x) \]

where
\[ R_n(0.1) = -\frac{1}{n+1} \frac{1}{(1-0.1)^{n+1}} \frac{1}{10^{n+1}} \]

for some \( z \) between 0 and 0.1. So
\[ |R_n(0.1)| \leq \frac{1}{n+1} \frac{1}{(1-0.1)^{n+1}} \frac{1}{10^{n+1}} = \frac{1}{n+1} \frac{1}{9^{n+1}} \]

When \( n = 4 \)
\[ |R_4(0.1)| \leq \frac{1}{5} \frac{1}{9^5} = 3.4 \times 10^{-6} \]

Hence
\[ \ln(0.9) = -\frac{1}{10} - \frac{1}{2} \frac{1}{10^2} - \frac{1}{3} \frac{1}{10^3} - \frac{1}{4} \frac{1}{10^4} \pm 3.4 \times 10^{-6} = -0.105358 \pm 5 \times 10^{-6} \]

83) Use a suitable series expansion to calculate
\[ \int_0^{1/4} e^{t^2} dt \]
to within \( 10^{-5} \).

**Solution.** Since every derivative of \( e^x \) is also \( e^x \)
\[ e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + R_n(x) \]

where
\[ R_n(x) = \frac{1}{(n+1)!}e^z x^{n+1} \]

for some \( z \) between 0 and \( x \). In particular, with \( 0 \leq x = t^2 \leq \frac{1}{4} \),
\[ e^{t^2} = 1 + t^2 + \frac{1}{2} t^4 + \frac{1}{3!} t^6 + \cdots + \frac{1}{n!} t^{2n} + R_n(t^2) \]

where
\[ R_n(t^2) = \frac{1}{(n+1)!}e^z t^{2(n+1)} \leq \frac{1}{(n+1)!}e^{1/16} \frac{1}{4^{2n+1}} \]

since \( z \) is between 0 and \( \frac{1}{16} \) and \( t \) is between 0 and \( \frac{1}{4} \). When \( n = 3 \)
\[ |R_n(t^2)| \leq \frac{1}{24} e^{1/16} \frac{1}{16} = 7 \times 10^{-7} \]
Hence

\[
\int_0^{1/4} e^{t^2} \, dt = \int_0^{1/4} \left[ 1 + t^2 + \frac{1}{2} t^4 + \frac{1}{3!} t^6 + 7 \times 10^{-7} \right] \, dt
\]

\[
= \left[ t + \frac{1}{3} t^3 + \frac{1}{10} t^5 + \frac{1}{42} t^7 + 7 \times 10^{-7} t \right]^{1/4}_0
\]

\[
= \frac{1}{4} + \frac{1}{3 \times 4 \pi} + \frac{1}{10 \times 4 \pi} + \frac{1}{42 \times 4 \pi} \pm 2 \times 10^{-7} = 0.255307 \pm 2 \times 10^{-6}
\]

84 a) Show that the series \( \sum_{n=1}^{\infty} \frac{1}{n^{4/5}} \) converges.

b) How many terms would be needed to compute the sum with an error less than 0.001?

c) Does the series \( \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1} \) converge? Justify your answer.

**Solution.**

a) Observe that \( \frac{1}{n^{4/5}} \leq \frac{1}{x^{4/5}} \) for all \( x \leq n \). In particular, \( \frac{1}{n^{4/5}} \leq \int_{n-1}^{n} \frac{1}{x^{4/5}} \, dx \). This forces

\[
\sum_{n=N}^{\infty} \frac{1}{n^{4/5}} \leq \sum_{n=N}^{\infty} \int_{n-1}^{n} \frac{1}{x^{4/5}} \, dx = \int_{N-1}^{\infty} \frac{1}{x^{4/5}} \, dx = \frac{x^{-3/5}}{-3/5} \bigg|_{N-1}^{\infty} = \frac{1}{3(N-1)^{3/5}}
\]

In particular \( S = 1 + \sum_{n=2}^{\infty} \frac{1}{n^{4/5}} \leq 1 + \frac{1}{3(2-1)^{3/5}} = \frac{4}{3} \), so that the series converges.

b) The tail \( \sum_{n=N+1}^{\infty} \frac{1}{n^{4/5}} < 0.001 \) if

\[
\frac{1}{3N^{3/5}} < 0.001 \iff N^{3/5} > \frac{1}{3 \times 0.001} \iff N > 3 \left( \frac{1}{0.003} \right)^{5/3} \iff N > 6.93
\]

If we keep at least 7 terms, the truncation error will be at most 0.001.

c) \( \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 1} \geq \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} > \int_{1}^{\infty} \frac{1}{\sqrt{x}} \, dx = \lim_{T \to \infty} 2\sqrt{T} \bigg|_{1}^{T} = \infty
\]

so the series diverges.

85 For each of the following functions, find a power series representation of the function.

a) \( f(x) = \frac{1}{4+x^2} \)

b) \( f(x) = \frac{1+x^2}{1-x} \)

c) \( f(x) = \frac{3x^2-2}{2x^2-3x+1} \)

d) \( f(x) = \ln\left( \frac{1+x}{1-x} \right) \)

e) \( f(x) = \frac{x^2}{(1-2x)^2} \)

**Solution.**

a) \( f(x) = \frac{1}{4+x^2} = \frac{1}{4} \left( \frac{1}{1-y} \right) \bigg|_{y=x^2/4} = \frac{1}{4} \sum_{n=0}^{\infty} y^n \bigg|_{y=-x^2/4} = \frac{1}{4} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{4^n} \)
b) 
\[ f(x) = \frac{1+x^2}{1-x^2} = (1 + x^2) \frac{1}{1-y} \bigg|_{y=x^2} = (1 + x^2) \sum_{n=0}^{\infty} y^n \bigg|_{y=x^2} = (1 + x^2) \sum_{n=0}^{\infty} x^{2n} \]

Combining

\[ \sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + x^6 + \ldots \]
\[ x^2 \sum_{n=0}^{\infty} x^{2n} = x^2 + x^4 + x^6 + \ldots \]

\[ \Rightarrow f(x) = 1 + 2x^2 + 2x^4 + 2x^6 + \ldots = 1 + \sum_{n=1}^{\infty} 2x^{2n} \]

c) 
\[ f(x) = \frac{3x-2}{2x^2-3x+1} = \frac{3x-2}{(1-2x)(1-x)} = -\frac{1}{1-2x} - \frac{1}{1-x} = -\sum_{n=0}^{\infty} (2x)^n - \sum_{n=0}^{\infty} x^n \]

\[ = -\sum_{n=0}^{\infty} (1 + 2^n) x^n \]

d) We start with

\[ \ln(1 + x) = \int_0^x \frac{1}{1+t} dt = \int_0^x [1 - t + t^2 - t^3 - \ldots] dt = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - \ldots \]

So

\[ f(x) = \ln(1 + x) - \ln(1 - x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - \ldots - \left[ -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots \right] \]

\[ = 2x + 2\frac{x^3}{3} + 2\frac{x^5}{5} + \ldots = \sum_{n=1}^{\infty} 2\frac{x^{2n+1}}{2n+1} \]

e) 
\[ f(x) = \frac{x^2}{(1-2x)^2} = \frac{x^2}{2} \frac{d}{dx} \frac{1}{1-2x} = \frac{x^2}{2} \frac{d}{dx} [1 + 2x + 2^2 x^2 + 2^3 x^3 + \ldots + 2^n x^n + \ldots] \]

\[ = \frac{x^2}{2} [2 + 2(2^2)x + 3(2^3)x^2 + \ldots + n(2^n)x^{n-1} + \ldots] \]

\[ = x^2 + 2^2 x^3 + 3(2^3)x^4 + \ldots + n(2^{n-1})x^{n+1} + \ldots \]

\[ = \sum_{n=1}^{\infty} n(2^{n-1})x^{n+1} \]