Trig Identities – Cosine Law and Addition Formulae

The cosine law

If a triangle has sides of length $A$, $B$ and $C$ and the angle opposite the side of length $C$ is $\theta$, then

$$C^2 = A^2 + B^2 - 2AB \cos \theta$$

Proof: Applying Pythagorous to the right hand triangle of the right hand figure gives

$$C^2 = (B - A \cos \theta)^2 + (A \sin \theta)^2$$

$$= B^2 - 2AB \cos \theta + A^2 \cos^2 \theta + A^2 \sin^2 \theta$$

$$= B^2 - 2AB \cos \theta + A^2$$

Addition and subtraction formulae

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$
$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$
$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$
$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$
**Proof:** We first prove \( \cos(x - y) = \cos x \cos y + \sin x \sin y \). The angle, of the upper triangle, that is opposite the side of length \( C \) is \( x - y \). So, by the cosine law,

\[
C^2 = 1^2 + 1^2 - 2 \times 1 \times 1 \times \cos(x - y) = 2 - 2 \cos(x - y)
\]

But the side of length \( C \) joins the points \((\cos y, \sin y)\) and \((\cos x, \sin x)\) and so we also have, by Pythagorous,

\[
C^2 = (\cos y - \cos x)^2 + (\sin y - \sin x)^2
\]

\[
= \cos^2 y - 2 \cos x \cos y + \cos^2 x + \sin^2 y - 2 \sin x \sin y + \sin^2 x
\]

\[
= 2 - 2 \cos x \cos y - 2 \sin x \sin y
\]

Setting the two formulae for \( C^2 \) equal to each other gives

\[
2 - 2 \cos(x - y) = 2 - 2 \cos x \cos y - 2 \sin x \sin y
\]

\[
\implies -2 \cos(x - y) = -2 \cos x \cos y - 2 \sin x \sin y
\]

\[
\implies \cos(x - y) = \cos x \cos y + \sin x \sin y
\]

which is the fourth addition formula. Replacing \( y \) by \(-y\) gives

\[
\cos(x + y) = \cos x \cos(-y) + \sin x \sin(-y) = \cos x \cos y - \sin x \sin y
\]

which is the third addition formula. Now, replacing \( x \) by \( \frac{\pi}{2} - x \) gives

\[
\cos \left( \frac{\pi}{2} - x + y \right) = \cos \left( \frac{\pi}{2} - x \right) \cos y - \sin \left( \frac{\pi}{2} - x \right) \sin y
\]

Recalling that \( \sin \left( \frac{\pi}{2} - z \right) = \cos z \) and \( \cos \left( \frac{\pi}{2} - z \right) = \sin z \),

\[
\sin(x - y) = \sin x \cos y - \cos x \sin y
\]

which is the second addition formula. Finally, replacing \( y \) by \(-y\) gives the first addition formula.

\[\square\]