Trig Functions

Definitions

\[
\begin{align*}
\sin \theta &= \frac{A}{C} & \cos \theta &= \frac{B}{C} & \tan \theta &= \frac{A}{B} \\
\csc \theta &= \frac{C}{A} & \sec \theta &= \frac{C}{B} & \cot \theta &= \frac{B}{A}
\end{align*}
\]

Radians

For use in calculus, angles are best measured in units called radians. By definition, an arc of length \( \theta \) on a circle of radius one subtends an angle of \( \theta \) radians at the center of the circle. Because the circumference of a circle of radius one is \( 2\pi \), we have

\[
2\pi \text{ radians} = 360^\circ \quad \pi \text{ radians} = 180^\circ \quad \frac{\pi}{2} \text{ radians} = 90^\circ \\
\frac{\pi}{3} \text{ radians} = 60^\circ \quad \frac{\pi}{4} \text{ radians} = 45^\circ \quad \frac{\pi}{6} \text{ radians} = 30^\circ
\]

Special Triangles

From the triangles above, we have

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( \sin \theta )</th>
<th>( \cos \theta )</th>
<th>( \tan \theta )</th>
<th>( \csc \theta )</th>
<th>( \sec \theta )</th>
<th>( \cot \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0^\circ = 0 \text{ rad} )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( 30^\circ = \frac{\pi}{6} \text{ rad} )</td>
<td>( \frac{1}{2} )</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>( \frac{1}{\sqrt{3}} )</td>
<td>2</td>
<td>( \frac{2}{\sqrt{3}} )</td>
<td>( \sqrt{3} )</td>
</tr>
<tr>
<td>( 45^\circ = \frac{\pi}{4} \text{ rad} )</td>
<td>( \frac{1}{\sqrt{2}} )</td>
<td>( \frac{1}{\sqrt{2}} )</td>
<td>1</td>
<td>( \sqrt{2} )</td>
<td>( \sqrt{2} )</td>
<td>1</td>
</tr>
<tr>
<td>( 60^\circ = \frac{\pi}{3} \text{ rad} )</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>1</td>
<td>( \frac{1}{\sqrt{3}} )</td>
<td>( \frac{\sqrt{3}}{2} )</td>
<td>2</td>
<td>( \frac{1}{\sqrt{3}} )</td>
</tr>
<tr>
<td>( 90^\circ = \frac{\pi}{2} \text{ rad} )</td>
<td>1</td>
<td>0</td>
<td></td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( 180^\circ = \pi \text{ rad} )</td>
<td>0</td>
<td>–1</td>
<td>0</td>
<td>–1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The empty boxes mean that the trig function is undefined (i.e. \( \pm\infty \)) for that angle.
Trig Identities – Elementary

The following identities are all immediate consequences of the definitions at the top of the previous page

\[
\begin{align*}
\csc \theta &= \frac{1}{\sin \theta} & \sec \theta &= \frac{1}{\cos \theta} & \tan \theta &= \frac{\sin \theta}{\cos \theta} & \cot \theta &= \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta}
\end{align*}
\]

Because \(2\pi\) radians is \(360^\circ\), the angles \(\theta\) and \(\theta + 2\pi\) are really the same, so

\[
\sin(\theta + 2\pi) = \sin \theta \quad \cos(\theta + 2\pi) = \cos \theta
\]

The following trig identities are consequences of the figure to their right.

\[
\begin{align*}
\sin^2 \theta + \cos^2 \theta &= 1 \\
\sin(-\theta) &= -\sin(\theta) \quad \cos(-\theta) &= \cos(\theta)
\end{align*}
\]

The following trig identities are consequences of the figure to their left.

\[
\begin{align*}
\sin \left(\frac{\pi}{2} - \theta\right) &= \cos \theta \\
\cos \left(\frac{\pi}{2} - \theta\right) &= \sin \theta
\end{align*}
\]

Trig Identities – Addition Formulae

The following trig identities are derived in the handout entitled “Trig Identities – Cosine law and Addition Formulae”

\[
\begin{align*}
\sin(x + y) &= \sin x \cos y + \cos x \sin y \\
\sin(x - y) &= \sin x \cos y - \cos x \sin y \\
\cos(x + y) &= \cos x \cos y - \sin x \sin y \\
\cos(x - y) &= \cos x \cos y + \sin x \sin y
\end{align*}
\]

Setting \(y = x\) gives

\[
\begin{align*}
\sin(2x) &= 2\sin x \cos x \\
\cos(2x) &= \cos^2 x - \sin^2 x \\
&= 2\cos^2 x - 1 \quad \text{since} \quad \sin^2 x = 1 - \cos^2 x \\
&= 1 - 2\sin^2 x \quad \text{since} \quad \cos^2 x = 1 - \sin^2 x
\end{align*}
\]

Solving \(\cos(2x) = 2\cos^2 x - 1\) for \(\cos^2 x\) and \(\cos(2x) = 1 - 2\sin^2 x\) for \(\sin^2 x\) gives

\[
\begin{align*}
\cos^2 x &= \frac{1 + \cos(2x)}{2} \\
\sin^2 x &= \frac{1 - \cos(2x)}{2}
\end{align*}
\]