

Error Formulae for Taylor Polynomial Approximations

Let

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{1}{n!}f^{(n)}(x_0)(x - x_0)^n$$

be the Taylor polynomial of degree n for the function $f(x)$ and expansion point x_0 . Using this polynomial to approximate $f(x)$ introduces an error

$$E_n(x) = f(x) - P_n(x)$$

We shall now prove that

$$E_n(x) = \frac{1}{(n+1)!}f^{(n+1)}(z)(x - x_0)^{n+1} \quad (1_n)$$

for some z between x_0 and x and that

$$E_n(x) = \frac{1}{n!} \int_{x_0}^x (x - z)^n f^{(n+1)}(z) dz \quad (2_n)$$

These proofs are not part of the official course. It rarely necessary, or even possible, to evaluate $E_n(x)$ exactly. It is usually sufficient to find a number M such that $|f^{(n+1)}(z)| \leq M$ for all z between x_0 and the x of interest. Both (1_n) and (2_n) then imply that $|E_n(x)| \leq \frac{1}{(n+1)!}M|x - x_0|^{n+1}$.

Both (1_n) and (2_n) are easily proven in the special case $n = 0$. When $n = 0$, (1_n) and (2_n) are the statements that

$$f(x) - f(x_0) = f'(z)(x - x_0) \quad (1_0)$$

for some z between x_0 and x and that

$$f(x) - f(x_0) = \int_{x_0}^x f'(z) dz \quad (2_0)$$

So (1₀) is just a restatement of the mean-value theorem and (2₀) is just a restatement of part of the fundamental theorem of calculus.

To prove (1_n) with $n \geq 1$, we need the following small generalization of the mean-value theorem.

Theorem (Generalized Mean-Value Theorem) *Let the functions $F(x)$ and $G(x)$ both be defined and continuous on $a \leq x \leq b$ and both be differentiable on $a < x < b$. Furthermore, suppose that $G'(x) \neq 0$ for all $a < x < b$. Then, there is a number c obeying $a < c < b$ such that*

$$\frac{F(b) - F(a)}{G(b) - G(a)} = \frac{F'(c)}{G'(c)}$$

Proof: Define

$$h(x) = [F(b) - F(a)][G(x) - G(a)] - [F(x) - F(a)][G(b) - G(a)]$$

Observe that $h(a) = h(b) = 0$. So, by the mean-value theorem, there is a number c obeying $a < c < b$ such that

$$0 = h'(c) = [F(b) - F(a)]G'(c) - F'(c)[G(b) - G(a)]$$

As $G(a) \neq G(b)$ (otherwise the mean-value theorem would imply the existence of an $a < x < b$ obeying $G'(x) = 0$), we may divide by $G'(c)[G(b) - G(a)]$ which gives the desired result. ■

Proof of (1_n): To prove (1₁), that is (1_n) for $n = 1$, simply apply the generalized mean-value theorem with $F(x) = f(x) - f(x_0) - f'(x_0)(x - x_0)$, $G(x) = (x - x_0)^2$, $a = x_0$ and $b = x$. Then $F(a) = G(a) = 0$, so that

$$\frac{F(b)}{G(b)} = \frac{F'(c)}{G'(c)} \Rightarrow \frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{(x - x_0)^2} = \frac{f'(c) - f'(x_0)}{2(c - x_0)}$$

for some c between x_0 and x . By the mean-value theorem (the standard one, but with $f(x)$ replaced by $f'(x)$), $\frac{f'(c) - f'(x_0)}{c - x_0} = f''(z)$, for some z between x_0 and c (which forces z to also be between x_0 and x). Hence

$$\frac{f(x) - f(x_0) - f'(x_0)(x - x_0)}{(x - x_0)^2} = \frac{1}{2} f''(z)$$

which is exactly (1₁).

At this stage, we know that (1_n) applies to all (sufficiently differentiable) functions for $n = 0$ and $n = 1$. To prove it for general n , we proceed by induction. That is, we assume that we already know that (1_n) applies to $n = k - 1$ for some k (as is the case for $k = 1, 2$) and that we wish to prove that it also applies to $n = k$. We apply the generalized mean-value theorem with $F(x) = E_k(x)$, $G(x) = (x - x_0)^{k+1}$, $a = x_0$ and $b = x$. Then $F(a) = G(a) = 0$, so that

$$\frac{F(b)}{G(b)} = \frac{F'(c)}{G'(c)} \Rightarrow \frac{E_k(x)}{(x - x_0)^{k+1}} = \frac{E'_k(c)}{(k+1)(c - x_0)^k}$$

But

$$\begin{aligned} E'_k(c) &= \frac{d}{dx} \left[f(x) - f(x_0) - f'(x_0) - \dots - \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k \right]_{x=c} \\ &= \left[f'(x) - f'(x_0) - \dots - \frac{1}{(k-1)!} f^{(k)}(x_0)(x - x_0)^{k-1} \right]_{x=c} \\ &= f'(c) - f'(x_0) - \dots - \frac{1}{(k-1)!} f^{(k)}(x_0)(c - x_0)^{k-1} \end{aligned}$$

The last expression is exactly the definition of $E_{k-1}(c)$, but for the function $f'(x)$, instead of the function $f(x)$. But we already know that (1_{k-1}) is true, so we already know that the last expression equals

$$\frac{1}{(k-1+1)!} (f')^{(k-1+1)}(z)(c - x_0)^{k-1+1} = \frac{1}{k!} f^{(k+1)}(z)(c - x_0)^k$$

for some z between x_0 and c . Subbing this in

$$\frac{E_k(x)}{(x - x_0)^{k+1}} = \frac{E'_k(c)}{(k+1)(c - x_0)^k} = \frac{1}{(k+1)!} f^{(k+1)}(z)$$

which is exactly (1_k). Repeating this for $k = 2, 3, 4, \dots$ gives (1_k) for all k . ■

Proof of (2_n): We again proceed by induction. That is, we assume that we already know that (2_n) applies to $n = k - 1$ for some k (as is the case for $k = 1$) and we then prove that it also applies to $n = k$. So we are assuming that

$$E_{k-1}(x) = \frac{1}{(k-1)!} \int_{x_0}^x (x - z)^{k-1} f^{(k)}(z) dz$$

Integrate by parts with $u(z) = f^{(k)}(z)$ and $v'(z) dz = \frac{(x - z)^{k-1}}{(k-1)!} dz$. Note that z is now the integration variable and x is just some constant. So $u'(z) dz = f^{(k+1)}(z) dz$ and we may take $v(z) = -\frac{1}{k!} (x - z)^k$. This gives

$$\begin{aligned} E_{k-1}(x) &= -\frac{1}{k!} (x - z)^k f^{(k)}(z) \Big|_{z=x_0}^{z=x} + \frac{1}{k!} \int_{x_0}^x (x - z)^k f^{(k+1)}(z) dz \\ &= \frac{1}{k!} (x - x_0)^k f^{(k)}(x_0) + \frac{1}{k!} \int_{x_0}^x (x - z)^k f^{(k+1)}(z) dz \end{aligned}$$

Since

$$E_k(x) = f(x) - P_k(x) = f(x) - P_{k-1}(x) - \frac{1}{k!} (x - x_0)^k f^{(k)}(x_0) = E_{k-1}(x) - \frac{1}{k!} (x - x_0)^k f^{(k)}(x_0)$$

we have

$$E_k(x) = \frac{1}{k!} \int_{x_0}^x (x - z)^k f^{(k+1)}(z) dz$$

which is exactly (2_k). Repeating this for $k = 2, 3, 4, \dots$ gives (2_k) for all k . ■