

# Substitution Examples

## Example 1

$$\int_0^{\pi/2} \cos(3x) dx = \int_0^{3\pi/2} \cos(y) \frac{dy}{3} \quad \begin{cases} y = 3x \\ dy = 3 dx \\ y(0) = 0 \\ y(\frac{\pi}{2}) = \frac{3}{2}\pi \end{cases}$$
$$= \frac{\sin y}{3} \Big|_0^{3\pi/2} = \frac{-1}{3} - \frac{0}{3} = \boxed{-\frac{1}{3}}$$

## Example 2

$$\int_0^1 \frac{1}{(2x+1)^3} dx = \int_1^3 \frac{1}{y^3} \frac{dy}{2} \quad \begin{cases} y = 2x + 1 \\ dy = 2 dx \\ y(0) = 1 \\ y(1) = 3 \end{cases}$$
$$= \frac{1}{2} \int_1^3 y^{-3} dy = \frac{1}{2} \frac{y^{-2}}{-2} \Big|_1^3 = \frac{3^{-2}}{-4} - \frac{1^{-2}}{-4} = \frac{1}{4} \left[ 1 - \frac{1}{9} \right] = \boxed{\frac{2}{9}}$$

## Example 3

$$\int_0^1 \frac{x}{1+x^2} dx = \int_1^2 \frac{1}{y} \frac{dy}{2} \quad \begin{cases} y = 1 + x^2 \\ dy = 2x dx \\ y(0) = 1 \\ y(1) = 2 \end{cases}$$
$$= \frac{1}{2} \ln |y| \Big|_1^2 = \frac{\ln 2}{2} - \frac{0}{2} = \boxed{\frac{1}{2} \ln 2}$$

### Example 4

$$\begin{aligned}\int \tan x \, dx &= \int \frac{1}{\cos x} \sin x \, dx = \int \frac{1}{y} \frac{dy}{-1} \quad \begin{cases} y = \cos x \\ dy = -\sin x \, dx \end{cases} \\ &= -\ln |y| + C = -\ln |\cos x| + C = \ln |\cos x|^{-1} + C \\ &= \boxed{\ln |\sec x| + C}\end{aligned}$$

$$\begin{aligned}\int \sec x \tan x \, dx &= \int \frac{1}{\cos^2 x} \sin x \, dx = \int \frac{1}{y^2} \frac{dy}{-1} \quad \begin{cases} y = \cos x \\ dy = -\sin x \, dx \end{cases} \\ &= -\frac{y^{-1}}{-1} + C = \frac{1}{y} + C = \boxed{\sec x + C}\end{aligned}$$

### Example 5

$$\begin{aligned}\int x^3 \cos(x^4 + 2) \, dx &= \int \cos(y) \frac{dy}{4} \quad \begin{cases} y = x^4 + 2 \\ dy = 4x^3 \, dx \end{cases} \\ &= \frac{1}{4} \sin y + C = \boxed{\frac{1}{4} \sin(x^4 + 2) + C}\end{aligned}$$

### Example 6

$$\begin{aligned}\int \sqrt{1+x^2} x^3 \, dx &= \int \sqrt{1+x^2} x^2 x \, dx \\ &= \int \sqrt{y} (y-1) \frac{dy}{2} \quad \begin{cases} y = 1+x^2 \\ dy = 2x \, dx \end{cases} \\ &= \frac{1}{2} \int (y^{3/2} - y^{1/2}) \, dy = \frac{1}{2} \left[ \frac{y^{5/2}}{5/2} - \frac{y^{3/2}}{3/2} \right] + C \\ &= \boxed{\frac{1}{5}(1+x^2)^{5/2} - \frac{1}{3}(1+x^2)^{3/2} + C}\end{aligned}$$

### Example 7

$$\int_0^1 \frac{1}{x^2 + 2x + 1} dx = \int_0^1 \frac{1}{(x+1)^2} dx = \int_1^2 \frac{1}{y^2} dy \quad \begin{cases} y = x + 1 \\ dy = dx \\ y(0) = 1 \\ y(1) = 2 \end{cases}$$
$$= \frac{y^{-1}}{-1} \Big|_1^2 = -\frac{1}{2} - (-1) = \boxed{\frac{1}{2}}$$

$$\int_0^1 \frac{1}{x^2 + 2x + 2} dx = \int_0^1 \frac{1}{(x+1)^2 + 1} dx = \int_1^2 \frac{1}{y^2 + 1} dy \quad \begin{cases} y = x + 1 \\ dy = dx \\ y(0) = 1 \\ y(1) = 2 \end{cases}$$
$$= \arctan y \Big|_1^2 = \boxed{\arctan 2 - \arctan 1}$$

$$\int_0^1 \frac{1}{x^2 + 2x + 5} dx = \int_0^1 \frac{1}{(x+1)^2 + 4} dx = \int_{\frac{1}{2}}^1 \frac{2dy}{4y^2 + 4} \quad \begin{cases} 2y = x + 1 \\ 2dy = dx \\ y(0) = 1/2 \\ y(1) = 1 \end{cases}$$
$$= \frac{1}{2} \arctan y \Big|_{1/2}^1 = \boxed{\frac{1}{2} [\arctan 1 - \arctan \frac{1}{2}]}$$

### Example 8

$$\int_0^a x \sqrt{a^2 - x^2} dx = \int_{a^2}^0 \sqrt{y} \frac{dy}{-2} \quad \begin{cases} y = a^2 - x^2 \\ dy = -2x dx \\ y(0) = a^2 \\ y(a) = 0 \end{cases}$$
$$= -\frac{1}{2} \frac{y^{3/2}}{3/2} \Big|_{a^2}^0 = \frac{1}{3} (a^2)^{3/2} = \boxed{\frac{1}{3} a^3}$$

### Example 9

$$\int \sqrt[3]{1-x} dx = \int \sqrt[3]{y} \frac{dy}{-1} \quad \begin{cases} y = 1 - x \\ dy = -dx \end{cases}$$
$$= -\frac{y^{4/3}}{4/3} + C = \boxed{-\frac{3}{4} (1-x)^{4/3} + C}$$

### Example 10

$$\begin{aligned}\int \sin(e^x) e^x dx &= \int \sin y dy \quad \begin{cases} y = e^x \\ dy = e^x dx \end{cases} \\ &= -\cos y + C = \boxed{-\cos(e^x) + C}\end{aligned}$$

### Example 11

$$\begin{aligned}\int x^3(1-x^2)^{3/2} dx &= \int (1-x^2)^{3/2} x^2 x dx \\ &= \int y^{3/2} (1-y) \frac{dy}{-2} \quad \begin{cases} y = 1-x^2 \\ dy = -2x dx \end{cases} \\ &= -\frac{1}{2} \int (y^{3/2} - y^{5/2}) dy = -\frac{1}{2} \left[ \frac{y^{5/2}}{5/2} - \frac{y^{7/2}}{7/2} \right] + C \\ &= \boxed{\frac{1}{7}(1-x^2)^{7/2} - \frac{1}{5}(1-x^2)^{5/2} + C}\end{aligned}$$

### Example 12

$$\begin{aligned}\int \frac{\cos x}{4 + \sin^2 x} dx &= \int \frac{2dy}{4 + 4y^2} \quad \begin{cases} 2y = \sin x \\ 2dy = \cos x dx \end{cases} \\ &= \frac{1}{2} \int \frac{dy}{1+y^2} = \frac{1}{2} \arctan y + C \\ &= \boxed{\frac{1}{2} \arctan \left( \frac{1}{2} \sin x \right) + C}\end{aligned}$$

### Example 13

$$\begin{aligned} \int \frac{\sin^3(\ln x) \cos^3(\ln x)}{x} dx &= \int \sin^3 y \cos^3 y dy \quad \begin{cases} y = \ln x \\ dy = \frac{dx}{x} \end{cases} \\ &= \int \sin^3 y (1 - \sin^2 y) \cos y dy \\ &= \int u^3 (1 - u^2) du \quad \begin{cases} u = \sin y \\ du = \cos y dy \end{cases} \\ &= \int [u^3 - u^5] du = \left[ \frac{u^4}{4} - \frac{u^6}{6} + C \right] \Big|_{u=\sin y} \Big|_{y=\ln x} \\ &= \boxed{\frac{\sin^4(\ln x)}{4} - \frac{\sin^6(\ln x)}{6} + C} \end{aligned}$$

It is always a good idea to check that an indefinite integral is correct by checking that its derivative is the integrand.

$$\begin{aligned} \frac{d}{dx} \left[ \frac{\sin^4(\ln x)}{4} - \frac{\sin^6(\ln x)}{6} \right] &= \sin^3(\ln x) \cos(\ln x) \frac{1}{x} - \sin^5(\ln x) \cos(\ln x) \frac{1}{x} \\ &= \sin^3(\ln x) \cos(\ln x) \frac{1}{x} [1 - \sin^2(\ln x)] \\ &= \sin^3(\ln x) \cos^3(\ln x) \frac{1}{x} \end{aligned}$$

There are often many different correct ways to evaluate an integral. Here is a second evaluation of the same indefinite integral that uses  $\sin(2y) = 2 \sin y \cos y$ .

$$\begin{aligned} \int \frac{\sin^3(\ln x) \cos^3(\ln x)}{x} dx &= \int \sin^3 y \cos^3 y dy \quad \begin{cases} y = \ln x \\ dy = \frac{dx}{x} \end{cases} \\ &= \frac{1}{8} \int \sin^3(2y) dy = \frac{1}{8} \int (1 - \cos^2(2y)) \sin(2y) dy \\ &= \frac{1}{8} \int [1 - z^2] \frac{dz}{-2} \quad \begin{cases} z = \cos(2y) \\ dz = -2 \sin(2y) dy \end{cases} \\ &= -\frac{1}{16} \left[ z - \frac{z^3}{3} + C \right] \Big|_{z=\cos(2y)} \Big|_{y=\ln x} \\ &= \boxed{-\frac{1}{16} \left[ \cos(2 \ln x) - \frac{1}{3} \cos^3(2 \ln x) \right] + C} \end{aligned}$$

We can also check that this answer is correct by differentiating:

$$\begin{aligned}
 & -\frac{1}{16} \frac{d}{dx} \left[ \cos(2 \ln x) - \frac{1}{3} \cos^3(2 \ln x) \right] \\
 &= -\frac{1}{16} \left[ -\sin(2 \ln x) \frac{2}{x} + \cos^2(2 \ln x) \sin(2 \ln x) \frac{2}{x} \right] \\
 &= \frac{1}{16} \left[ 1 - \cos^2(2 \ln x) \right] \sin(2 \ln x) \frac{2}{x} \\
 &= \frac{1}{8} \sin^3(2 \ln x) \frac{1}{x} \\
 &= \sin^3(\ln x) \cos^3(\ln x) \frac{1}{x}
 \end{aligned}$$

We can also use the identities

$$\cos(2y) = 1 - 2 \sin^2 y \quad \text{and} \quad (a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

to check that the answers are the same

$$\begin{aligned}
 & -\frac{1}{16} \left[ \cos(2 \ln x) - \frac{1}{3} \cos^3(2 \ln x) \right] \\
 &= -\frac{1}{48} \left[ 3(1 - 2 \sin^2 y) - (1 - 2 \sin^2 y)^3 \right] \\
 &= -\frac{1}{48} \left[ 3 - 6 \sin^2 y - (1 - 6 \sin^2 y + 12 \sin^4 y - 8 \sin^6 y) \right] \\
 &= -\frac{1}{48} \left[ 2 - 12 \sin^4 y + 8 \sin^6 y \right] \\
 &= \frac{\sin^4(\ln x)}{4} - \frac{\sin^6(\ln x)}{6} - \frac{1}{24}
 \end{aligned}$$

The extra term  $\frac{1}{24}$  just means that the symbol  $C$  that appears in the two answers

$$\frac{\sin^4(\ln x)}{4} - \frac{\sin^6(\ln x)}{6} + C \quad \text{and} \quad -\frac{1}{16} \left[ \cos(2 \ln x) - \frac{1}{3} \cos^3(2 \ln x) \right] + C$$

actually stands two different constants.