

Substitution Examples

Example 1

$$\int_0^{\pi/2} \cos(3x) dx = \int_0^{3\pi/2} \cos(y) \frac{dy}{3} \quad \begin{cases} y=3x \\ dy=3 dx \\ y(0)=0 \\ y(\frac{\pi}{2})=\frac{3}{2}\pi \end{cases}$$
$$= \frac{\sin y}{3} \Big|_0^{3\pi/2} = \frac{-1}{3} - \frac{0}{3} = \boxed{-\frac{1}{3}}$$

Example 2

$$\int_0^1 \frac{1}{(2x+1)^3} dx = \int_1^3 \frac{1}{y^3} \frac{dy}{2} \quad \begin{cases} y=2x+1 \\ dy=2 dx \\ y(0)=1 \\ y(1)=3 \end{cases}$$
$$= \frac{1}{2} \int_1^3 y^{-3} dy = \frac{1}{2} \frac{y^{-2}}{-2} \Big|_1^3 = \frac{3^{-2}}{-4} - \frac{1^{-2}}{-4} = \frac{1}{4} [1 - \frac{1}{9}] = \boxed{\frac{2}{9}}$$

Example 3

$$\int_0^1 \frac{x}{1+x^2} dx = \int_1^2 \frac{1}{y} \frac{dy}{2} \quad \begin{cases} y=1+x^2 \\ dy=2x dx \\ y(0)=1 \\ y(1)=2 \end{cases}$$
$$= \frac{1}{2} \ln|y| \Big|_1^2 = \frac{\ln 2}{2} - \frac{0}{2} = \boxed{\frac{1}{2} \ln 2}$$

Example 4

$$\int \tan x dx = \int \frac{1}{\cos x} \sin x dx = \int \frac{1}{y} \frac{dy}{-1} \quad \begin{cases} y=\cos x \\ dy=-\sin x dx \end{cases}$$
$$= -\ln|y| + C = -\ln|\cos x| + C = \ln|\cos x|^{-1} + C$$
$$= \boxed{\ln|\sec x| + C}$$
$$\int \sec x \tan x dx = \int \frac{1}{\cos^2 x} \sin x dx = \int \frac{1}{y^2} \frac{dy}{-1} \quad \begin{cases} y=\cos x \\ dy=-\sin x dx \end{cases}$$
$$= -\frac{y^{-1}}{-1} + C = \frac{1}{y} + C = \boxed{\sec x + C}$$

Example 5

$$\begin{aligned} \int x^3 \cos(x^4 + 2) dx &= \int \cos(y) \frac{dy}{4} \quad \begin{cases} y = x^4 + 2 \\ dy = 4x^3 dx \end{cases} \\ &= \frac{1}{4} \sin y + C = \boxed{\frac{1}{4} \sin(x^4 + 2) + C} \end{aligned}$$

Example 6

$$\begin{aligned} \int \sqrt{1+x^2} x^3 dx &= \int \sqrt{1+x^2} x^2 x dx \\ &= \int \sqrt{y} (y-1) \frac{dy}{2} \quad \begin{cases} y = 1+x^2 \\ dy = 2x dx \end{cases} \\ &= \frac{1}{2} \int (y^{3/2} - y^{1/2}) dy = \frac{1}{2} \left[\frac{y^{5/2}}{5/2} - \frac{y^{3/2}}{3/2} \right] + C \\ &= \boxed{\frac{1}{5}(1+x^2)^{5/2} - \frac{1}{3}(1+x^2)^{3/2} + C} \end{aligned}$$

Example 7

$$\begin{aligned} \int_0^1 \frac{1}{x^2 + 2x + 1} dx &= \int_0^1 \frac{1}{(x+1)^2} dx = \int_1^2 \frac{1}{y^2} dy \quad \begin{cases} y = x + 1 \\ dy = dx \\ y(0) = 1 \\ y(1) = 2 \end{cases} \\ &= \left. \frac{y^{-1}}{-1} \right|_1^2 = -\frac{1}{2} - (-1) = \boxed{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned} \int_0^1 \frac{1}{x^2 + 2x + 2} dx &= \int_0^1 \frac{1}{(x+1)^2 + 1} dx = \int_1^2 \frac{1}{y^2 + 1} dy \quad \begin{cases} y = x + 1 \\ dy = dx \\ y(0) = 1 \\ y(1) = 2 \end{cases} \\ &= \arctan y \Big|_1^2 = \boxed{\arctan 2 - \arctan 1} \end{aligned}$$

$$\begin{aligned} \int_0^1 \frac{1}{x^2 + 2x + 5} dx &= \int_0^1 \frac{1}{(x+1)^2 + 4} dx = \int_{\frac{1}{2}}^1 \frac{2dy}{4y^2 + 4} \quad \begin{cases} 2y = x + 1 \\ 2dy = dx \\ y(0) = 1/2 \\ y(1) = 1 \end{cases} \\ &= \frac{1}{2} \arctan y \Big|_{1/2}^1 = \boxed{\frac{1}{2} [\arctan 1 - \arctan \frac{1}{2}]} \end{aligned}$$

Example 8

$$\int_0^a x \sqrt{a^2 - x^2} dx = \int_{a^2}^0 \sqrt{y} \frac{dy}{-2} \quad \begin{cases} y = a^2 - x^2 \\ dy = -2x dx \\ y(0) = a^2 \\ y(a) = 0 \end{cases}$$

$$= -\frac{1}{2} \frac{y^{3/2}}{3/2} \Big|_{a^2}^0 = \frac{1}{3} (a^2)^{3/2} = \boxed{\frac{1}{3} a^3}$$

Example 9

$$\int \sqrt[3]{1-x} dx = \int \sqrt[3]{y} \frac{dy}{-1} \quad \begin{cases} y = 1-x \\ dy = -dx \end{cases}$$

$$= -\frac{y^{4/3}}{4/3} + C = \boxed{-\frac{3}{4}(1-x)^{4/3} + C}$$

Example 10

$$\int \sin(e^x) e^x dx = \int \sin y dy \quad \begin{cases} y = e^x \\ dy = e^x dx \end{cases}$$

$$= -\cos y + C = \boxed{-\cos(e^x) + C}$$

Example 11

$$\int x^3(1-x^2)^{3/2} dx = \int (1-x^2)^{3/2} x^2 x dx$$

$$= \int y^{3/2} (1-y) \frac{dy}{-2} \quad \begin{cases} y = 1-x^2 \\ dy = -2x dx \end{cases}$$

$$= -\frac{1}{2} \int (y^{3/2} - y^{5/2}) dy = -\frac{1}{2} \left[\frac{y^{5/2}}{5/2} - \frac{y^{7/2}}{7/2} \right] + C$$

$$= \boxed{\frac{1}{7}(1-x^2)^{7/2} - \frac{1}{5}(1-x^2)^{5/2} + C}$$

Example 12

$$\int \frac{\cos x}{4 + \sin^2 x} dx = \int \frac{2dy}{4 + 4y^2} \quad \begin{cases} 2y = \sin x \\ 2dy = \cos x dx \end{cases}$$

$$= \frac{1}{2} \int \frac{dy}{1+y^2} = \frac{1}{2} \arctan y + C$$

$$= \boxed{\frac{1}{2} \arctan \left(\frac{1}{2} \sin x \right) + C}$$

Example 13

$$\begin{aligned}
\int \frac{\sin^3(\ln x) \cos^3(\ln x)}{x} dx &= \int \sin^3 y \cos^3 y dy \quad \begin{cases} y = \ln x \\ dy = \frac{dx}{x} \end{cases} \\
&= \int \sin^3 y (1 - \sin^2 y) \cos y dy \\
&= \int u^3 (1 - u^2) du \quad \begin{cases} u = \sin y \\ du = \cos y dy \end{cases} \\
&= \int [u^3 - u^5] du = \left[\frac{u^4}{4} - \frac{u^6}{6} + C \right] \Big|_{u=\sin y} \Big|_{y=\ln x} \\
&= \boxed{\frac{\sin^4(\ln x)}{4} - \frac{\sin^6(\ln x)}{6} + C}
\end{aligned}$$

It is always a good idea to check that an indefinite integral is correct by checking that its derivative is the integrand.

$$\begin{aligned}
\frac{d}{dx} \left[\frac{\sin^4(\ln x)}{4} - \frac{\sin^6(\ln x)}{6} \right] &= \sin^3(\ln x) \cos(\ln x) \frac{1}{x} - \sin^5(\ln x) \cos(\ln x) \frac{1}{x} \\
&= \sin^3(\ln x) \cos(\ln x) \frac{1}{x} [1 - \sin^2(\ln x)] \\
&= \sin^3(\ln x) \cos^3(\ln x) \frac{1}{x}
\end{aligned}$$

There are often many different correct ways to evaluate an integral. Here is a second evaluation of the same indefinite integral that uses $\sin(2y) = 2 \sin y \cos y$.

$$\begin{aligned}
\int \frac{\sin^3(\ln x) \cos^3(\ln x)}{x} dx &= \int \sin^3 y \cos^3 y dy \quad \begin{cases} y = \ln x \\ dy = \frac{dx}{x} \end{cases} \\
&= \frac{1}{8} \int \sin^3(2y) dy = \frac{1}{8} \int (1 - \cos^2(2y)) \sin(2y) dy \\
&= \frac{1}{8} \int [1 - z^2] \frac{dz}{-2} \quad \begin{cases} z = \cos(2y) \\ dz = -2 \sin(2y) dy \end{cases} \\
&= -\frac{1}{16} \left[z - \frac{z^3}{3} + C \right] \Big|_{z=\cos(2y)} \Big|_{y=\ln x} \\
&= \boxed{-\frac{1}{16} \left[\cos(2 \ln x) - \frac{1}{3} \cos^3(2 \ln x) \right] + C}
\end{aligned}$$

We can also check that this answer is correct by differentiating:

$$\begin{aligned}
&-\frac{1}{16} \frac{d}{dx} \left[\cos(2 \ln x) - \frac{1}{3} \cos^3(2 \ln x) \right] \\
&= -\frac{1}{16} \left[-\sin(2 \ln x) \frac{2}{x} + \cos^2(2 \ln x) \sin(2 \ln x) \frac{2}{x} \right] \\
&= \frac{1}{16} [1 - \cos^2(2 \ln x)] \sin(2 \ln x) \frac{2}{x} \\
&= \frac{1}{8} \sin^3(2 \ln x) \frac{1}{x} \\
&= \sin^3(\ln x) \cos^3(\ln x) \frac{1}{x}
\end{aligned}$$

We can also use the identities

$$\cos(2y) = 1 - 2\sin^2 y \quad \text{and} \quad (a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

to check that the answers are the same

$$\begin{aligned} & -\frac{1}{16} \left[\cos(2 \ln x) - \frac{1}{3} \cos^3(2 \ln x) \right] \\ &= -\frac{1}{48} \left[3(1 - 2\sin^2 y) - (1 - 2\sin^2 y)^3 \right] \\ &= -\frac{1}{48} \left[3 - 6\sin^2 y - (1 - 6\sin^2 y + 12\sin^4 y - 8\sin^6 y) \right] \\ &= -\frac{1}{48} \left[2 - 12\sin^4 y + 8\sin^6 y \right] \\ &= \frac{\sin^4(\ln x)}{4} - \frac{\sin^6(\ln x)}{6} - \frac{1}{24} \end{aligned}$$

The extra term $\frac{1}{24}$ just means that the symbol C that appears in the two answers

$$\frac{\sin^4(\ln x)}{4} - \frac{\sin^6(\ln x)}{6} + C \quad \text{and} \quad -\frac{1}{16} \left[\cos(2 \ln x) - \frac{1}{3} \cos^3(2 \ln x) \right] + C$$

actually stands two different constants.