Integration of $\sec x$ and $\sec^3 x$

$\int \sec x\ dx$ – by trickery

The standard trick used to integrate $\sec x$ is to multiply the integrand by $1 = \frac{\sec x+\tan x}{\sec x+\tan x}$ and then substitute $y = \sec x + \tan x$, $dy = (\sec x \tan x + \sec^2 x)\,dx$.

$$\begin{aligned}\int \sec x\ dx &= \int \sec x\ \tfrac{\sec x+\tan x}{\sec x+\tan x}\ dx = \int \tfrac{\sec^2 x+\sec x\tan x}{\sec x+\tan x}\ dx = \int \tfrac{dy}{y} = \ln|y| + C\\ &= \ln|\sec x + \tan x| + C\end{aligned}$$

$\int \sec x\ dx$ – by partial fractions

Another method for integrating $\int \sec x\ dx$, that is more tedious, but less dependent on a memorized trick, is to convert $\int \sec x\,dx$ into the integral of a rational function using the substitution $y = \sin x$, $dy = \cos x\,dx$ and then use partial fractions.

$$\begin{aligned}\int \sec x\ dx &= \int \tfrac{1}{\cos x}\ dx = \int \tfrac{\cos x}{\cos^2 x}\ dx = \int \tfrac{\cos x}{1-\sin^2 x}\ dx = \int \tfrac{1}{1-y^2}\ dy\\ &= \int \tfrac{1}{2}\Big[\tfrac{1}{1+y} + \tfrac{1}{1-y}\Big]\ dy = \tfrac{1}{2}\int \Big[\tfrac{1}{y+1} - \tfrac{1}{y-1}\Big]\ dy = \tfrac{1}{2}\Big[\ln|y+1| - \ln|y-1|\Big] + C\\ &= \tfrac{1}{2}\ln\left|\tfrac{y+1}{y-1}\right| + C = \tfrac{1}{2}\ln\left|\tfrac{\sin x+1}{\sin x-1}\right| + C\end{aligned}$$

To see that this answer is really the same as the one above, note that

$$\begin{aligned}&\tfrac{\sin x+1}{\sin x-1} = \tfrac{(\sin x+1)^2}{\sin^2 x-1} = \tfrac{(\sin x+1)^2}{-\cos^2 x}\\ \Rightarrow &\tfrac{1}{2}\ln\left|\tfrac{\sin x+1}{\sin x-1}\right| = \tfrac{1}{2}\ln\left|\tfrac{(\sin x+1)^2}{-\cos^2 x}\right| = \tfrac{1}{2}\ln\left|\tfrac{(\sin x+1)^2}{\cos^2 x}\right| = \ln\left|\tfrac{\sin x+1}{\cos x}\right| = \ln|\tan x + \sec x|\end{aligned}$$

$\int \sec^3 x\ dx$ – by trickery

The standard trick used to evaluate $\int \sec^3 x\ dx$ is integration by parts with $u = \sec x$, $dv = \sec^2 x\ dx$, $du = \sec x \tan x\ dx$, $v = \tan x$.

$$\int \sec^3 x\ dx = \int \sec x\ \sec^2 x\ dx = \sec x \tan x - \int \tan x\ \sec x \tan x\ dx$$

Since $\tan^2 x + 1 = \sec^2 x$, we have $\tan^2 x = \sec^2 x - 1$ and

$$\int \sec^3 x\,dx = \sec x\tan x - \int [\sec^3 x - \sec x]\,dx = \sec x \tan x + \ln|\sec x + \tan x| + C - \int \sec^3 x\,dx$$

where we used $\int \sec x\ dx = \ln|\sec x + \tan x| + C$. Now moving the $\int \sec^3 x\ dx$ from the right hand side to the left hand side

$$\begin{aligned}2\int \sec^3 x\ dx &= \sec x \tan x + \ln|\sec x + \tan x| + C\\ \Rightarrow \int \sec^3 x\ dx &= \tfrac{1}{2}\sec x \tan x + \tfrac{1}{2}\ln|\sec x + \tan x| + C\end{aligned}$$

for a new arbitrary constant C.

$\int \sec^3 x \ dx$ – by partial fractions

Another method for integrating $\int \sec^3 x \ dx$, that is more tedious, but less dependent on trickery, is to convert $\int \sec^3 x \, dx$ into the integral of a rational function using the substitution $y = \sin x$, $dy = \cos x \, dx$ and then use partial fractions.

$$\begin{aligned}
\int \sec^3 x \ dx &= \int \tfrac{1}{\cos^3 x} \ dx = \int \tfrac{\cos x}{\cos^4 x} \ dx = \int \tfrac{\cos x}{[1-\sin^2 x]^2} \ dx = \int \tfrac{1}{[1-y^2]^2} \ dy = \int \tfrac{1}{[y^2-1]^2} \ dy \\
&= \int \Big[\tfrac{1}{2}\big(\tfrac{1}{y-1} - \tfrac{1}{y+1}\big)\Big]^2 \ dy = \tfrac{1}{4} \int \Big[\tfrac{1}{(y-1)^2} - \tfrac{2}{(y-1)(y+1)} + \tfrac{1}{(y+1)^2}\Big] \ dy \\
&= \tfrac{1}{4} \int \Big[\tfrac{1}{(y-1)^2} - \tfrac{1}{y-1} + \tfrac{1}{y+1} + \tfrac{1}{(y+1)^2}\Big] \ dy \\
&= \tfrac{1}{4}\Big[-\tfrac{1}{y-1} - \ln|y-1| + \ln|y+1| - \tfrac{1}{y+1}\Big] + C = -\tfrac{1}{4}\tfrac{2y}{y^2-1} + \tfrac{1}{4}\ln\big|\tfrac{y+1}{y-1}\big| + C \\
&= \tfrac{1}{2}\tfrac{y}{1-y^2} + \tfrac{1}{4}\ln\big|\tfrac{y+1}{y-1}\big| + C = \tfrac{1}{2}\tfrac{\sin x}{\cos^2 x} + \tfrac{1}{2}\ln\big|\tfrac{\sin x+1}{\sin x-1}\big| + C
\end{aligned}$$