

Richardson Extrapolation and Romberg Integration

There are many approximation procedures in which one first picks a step size h and then generates an approximation $A(h)$ to some desired quantity A . Often the order of the error generated by the procedure is known. This means

$$A = A(h) + Kh^k + K'h^{k+1} + K''h^{k+2} + \dots$$

with k being some known constant, called the order of the error, and K, K', K'', \dots being some other (usually unknown) constants. For example, A might be the value of some integral $\int_a^b f(x) dx$. For the Trapezoidal Rule with n steps, $\Delta x = \frac{b-a}{n}$ plays the role of the step size. If $A(h)$ is the approximation to A produced by Trapezoidal Rule with $\Delta x = h$, then $k = 2$. If Simpson's Rule is used, $k = 4$.

The notation $O(h^{k+1})$ is conventionally used to stand for "a sum of terms of order h^{k+1} and higher". So the above equation may be written

$$A = A(h) + Kh^k + O(h^{k+1}) \tag{1}$$

If we were to drop the, hopefully tiny, term $O(h^{k+1})$ from this equation, we would have one linear equation, $A = A(h) + Kh^k$, in the two unknowns A, K . But this really gives a different equation for each different value of h . We can get two different equations for A and K by just using two different step sizes. Then the two equations may be solved, yielding approximate values of A and K . We do this now, using step sizes h and $h/2$, for any h . Taking 2^k times

$$A = A(h/2) + K(h/2)^k + O(h^{k+1}) \tag{2}$$

(note that, in equations (1) and (2), the symbol " $O(h^{k+1})$ " is used to stand for two **different** sums of terms of order h^{k+1} and higher) and subtracting equation (1) gives

$$\begin{aligned} (2^k - 1)A &= 2^k A(h/2) - A(h) + O(h^{k+1}) \\ A &= \frac{2^k A(h/2) - A(h)}{2^k - 1} + O(h^{k+1}) \end{aligned}$$

Hence if we define

$$B(h) = \frac{2^k A(h/2) - A(h)}{2^k - 1} \tag{3}$$

then

$$A = B(h) + O(h^{k+1}) \tag{4}$$

and $B(h)$ is an approximation whose error is of order $k+1$, one better than $A(h)$'s. The generation of a "new improved" approximation for A from two $A(h)$'s with different values of h is called Richardson Extrapolation.

If $A(h)$ has been computed for three values of h , we can generate $B(h)$ for two values of h . If the order of the error in $B(h)$ is known, the above procedure can be repeated with a new value of k . And so on. One widely used numerical integration algorithm, called Romberg integration, applies this procedure repeatedly to the Trapezoidal Rule. It is known that the Trapezoidal Rule approximation $T(h)$ to an integral I has error behaviour (assuming that the integrand $f(x)$ is smooth)

$$I = T(h) + K_1 h^2 + K_2 h^4 + K_3 h^6 + \dots$$

Hence

$$T_1(h) = \frac{4T(h/2) - T(h)}{3} \quad \text{has error of order 4, so that}$$

$$T_2(h) = \frac{16T_1(h/2) - T_1(h)}{15} \quad \text{has error of order 6, so that}$$

$$T_3(h) = \frac{64T_2(h/2) - T_2(h)}{63} \quad \text{has error of order 8 and so on}$$

We know another method which produces an error of order 4 – Simpson’s Rule. In fact, $T_1(h)$ is exactly Simpson’s Rule (for step size $\frac{h}{2}$).

Example

$$A = \int_0^\pi \sin x \, dx = 2$$

h	$T(h)$	%	$T_1(h)$	%	$T_2(h)$	%	$T_3(h)$	%
$\pi/4$	1.896	5.2	2.0002692	-1.3×10^{-2}	1.999999752	1.2×10^{-5}	2.000000000060	-2.9×10^{-9}
$\pi/8$	1.974	1.3	2.0000166	-8.3×10^{-4}	1.999999996	1.9×10^{-7}		
$\pi/16$	1.993	.32	2.0000010	-5.2×10^{-5}				
$\pi/32$	1.998	.08						

The “%” column gives the percentage error.